Ejemplo:

a) Hallar por CM la solución de $Ax=b$ con $A=\begin{pmatrix}1&0&1\\0&0&0\\-1&1&0\end{pmatrix}$ $b=\begin{pmatrix}1\\1\\0\end{pmatrix}$ $\qquad C_1=-C_2+C_3 \quad C_1\ C_2\ C_3$

b) Buscar entre todas las soluciones el de norma mínima. $\rho(A)=2$

a) Hallar por CM la solución de $Ax=b$ $\quad A=\begin{pmatrix}1&1\\0&0\\-1&1\end{pmatrix}$ $\quad b=\begin{pmatrix}1\\1\\0\end{pmatrix}$

$Col(A)=\{x\in\mathbb{R}^3 : x_2=0\}$ $\qquad b\notin Col(A)$

Busco $A\hat{x} = P_{Col(A)}\, b = \begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow A\hat{x}=\begin{pmatrix}1\\0\\0\end{pmatrix}$

$\begin{pmatrix}1&1\\0&0\\-1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1\\0\\0\end{pmatrix}$ $\quad x_1+x_2=1 \quad -x_1+x_2=0 \quad x_1=x_2=1/2$

La solución por MC es única porque las columnas de A forman un g. L.I.

Resuelvo $A^TA\hat{x}=A^Tb$

$\begin{pmatrix}1&0&-1\\1&0&1\end{pmatrix}\begin{pmatrix}1&1\\0&0\\-1&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1&0&-1\\1&0&1\end{pmatrix}\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$

$\underbrace{\begin{pmatrix}2&0\\0&2\end{pmatrix}}_{\text{tiene inversa}}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix} \Rightarrow x_1=x_2=1/2$

$(A^TA)^{-1}=\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}$

$\hat{x}=(A^TA)^{-1}A^Tb$ $\qquad A\hat{x}=\underbrace{A(A^TA)^{-1}A^T}_{\text{proyecta sobre } Col(A)}b=Proy_{Col(A)}(b)$

$A(A^TA)^{-1}A^T=\underset{3\times2}{\begin{pmatrix}1&1\\0&0\\1&-1\end{pmatrix}}\underset{2\times2}{\begin{pmatrix}1/2&0\\0&1/2\end{pmatrix}}\underset{2\times3}{\begin{pmatrix}1&0&1\\1&0&-1\end{pmatrix}}=$

$=\begin{pmatrix}1/2&1/2\\0&0\\1/2&-1/2\end{pmatrix}\begin{pmatrix}1&0&1\\1&0&-1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$

matriz que proyecta sobre Col(A) en base canónica

$Col(A)=gen\{e_1,e_3\}$ $\qquad (Col(A))^\perp=gen\{e_2\}$

$A=\begin{pmatrix}1&-1&0\\1&0&1\\-2&1&-1\end{pmatrix}$ $\quad C_3=C_1+C_2$ $\qquad Col(A)=\{x\in\mathbb{R}^3: x+y+z=0\}$

$b\notin Col(A)$ $\qquad b=\begin{pmatrix}1\\1\\1\end{pmatrix}\in(Col(A))^\perp$

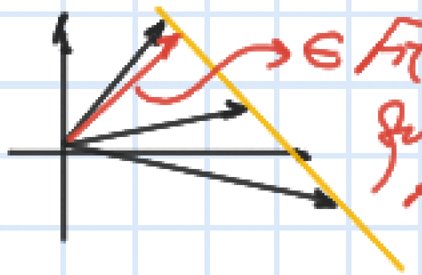
$(Col(A))^\perp=gen\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$

Resuelvo por CM

1) Calcula $\underset{Col(A)}{P}(b) = 0_{\mathbb{R}^3}$ o $\underset{(Col(A))^\perp}{P}(b) = b$

Resuelvo $A\hat{x}=0 \Rightarrow \hat{x}\in Nul(A)$ en este caso!

$\begin{pmatrix}1&-1&0\\1&0&1\\-2&1&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=0$



$\in Fil(A)$ ya que $Fil(A)=Nul(A)^\perp$

$x_1=x_2 \qquad x_1=-x_3$

$\hat{x}=\begin{pmatrix}x_1\\x_1\\-x_1\end{pmatrix}=\alpha\begin{pmatrix}1\\1\\-1\end{pmatrix}$ Cual sería la de norma mínima en este caso

$\hat{x}=\overset{0}{x_{Fil(A)}}+\overline{x_{Nul(A)}}$

La solución de norma mínima es $\hat{x}=0$ y su norma es 0

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & -1 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$C_3 = C_1 + C_2$

$$Col(A) = gen\left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} = gen\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$(Col(A))^{\perp} = gen\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

$$\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \underbrace{\overset{1/3}{\alpha} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \overset{-1/3}{\beta} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{\begin{pmatrix} -1/3 \\ -1/3 \\ 2/3 \end{pmatrix} \in Col(A)} + \overset{4/3}{\gamma} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\frac{4}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \in (Col(A))^{\perp}$$

$$\begin{cases} -\alpha + \gamma = 1 \\ \beta + \gamma = 1 \\ \alpha - \beta + \gamma = 1 \end{cases} \qquad \left(\begin{array}{ccc|c} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 1 & 2 \end{array}\right) F_3 + F_1$$

$$\left(\begin{array}{ccc|c} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 2 & 3 \end{array}\right) F_3 + F_2 \qquad \left(\begin{array}{ccc|c} -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 4 \end{array}\right)$$

$\gamma = 4/3 \qquad \beta + \gamma = 1 = \beta = -1/3$

$-\alpha + \gamma = 1 \qquad \alpha = 1/3$

$$A\hat{x} = \underset{Col(A)}{P}(b) = \begin{pmatrix} -1/3 \\ -1/3 \\ 2/3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ -2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1/3 \\ -1/3 \\ 2/3 \end{pmatrix}$$

$x_1 - x_2 = -1/3 \quad E_1$

$x_1 + x_3 = -1/3 \quad E_2$

$-2x_1 + x_2 - x_3 = 2/3 \quad -(E_1 + E_2)$

$x_1 = x_2 - 1/3 \qquad x_3 = -x_1 - 1/3$

$x_2 = x_2 \qquad x_3 = -x_2 - 2/3$

$$\hat{x} = \begin{pmatrix} x_2 - 1/3 \\ x_2 \\ -x_2 - 2/3 \end{pmatrix} = \underbrace{\begin{pmatrix} -1/3 \\ 0 \\ -2/3 \end{pmatrix}}_{\in Fil(A)?} + \underbrace{x_2 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}_{\in Nul(A)}$$

no

$$\|\hat{x}\|^2 = (x_2 - 1/3)^2 + x_2^2 + (-x_2 - 2/3)^2 = f(x_2)$$

$$f'(x_2) = 2(x_2 - 1/3) + 2x_2 + 2(x_2 + 2/3)$$

$6x_2 + 2/3 = 0 \qquad x_2 = -1/9$

$$x_2 = -1/9 \qquad \hat{x} = \underbrace{\begin{pmatrix} -4/9 \\ -1/9 \\ -5/9 \end{pmatrix}}_{\in Fil(A)} \perp \underbrace{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}}_{\in Nul(A)}$$