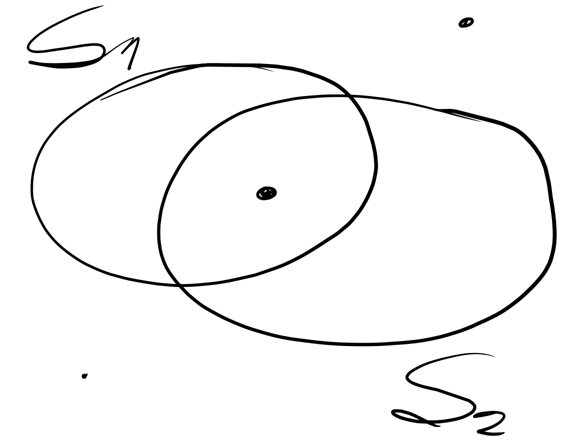
Encontrar T: $S_1 \oplus T = S_2 \oplus T = \mathbb{R}_3[x]$

$S_1 = \{ p \in \mathbb{R}_3[x] : p(1) = p(3) = 0 \}$

$S_2 = \{ p \in \mathbb{R}_3[x] \quad p(3) = p(5) = 0 \}$



$\dim S_1 = \dim S_2 = 2 \Rightarrow \dim T = 2$

$S_1 \cap S_2 = \{ p \in \mathbb{R}_3[x] : p(1) = p(3) = p(5) = 0 \}$

$\dim S_1 \cap S_2 = 1$

S_1: $p \in S_1 \quad p = a(x-1)(x-3)(x-b)$

$p \in S_2 \quad p = \alpha(x-3)(x-5)(x-\beta)$

$T_1 = \{ p \in \mathbb{R}_3[x] : p(0) = p(2) = 0 \}$

$T_2 = \{ \ldots\ldots \quad p(9) = p(11) = 0 \}$

$$T_1 = \{ p \in \mathbb{R}_3[x] : p(0) = p(2) = 0 \}$$

$$p \in T_1 \Rightarrow \quad p = m(x-2)\,x\,(x-r)$$
$$= m(x^2-2x)(x-r)$$
$$= m(x^3-2x^2) - mr(x^2-2x)$$

$\dim T_1 = 2$ $\qquad S_1 = \{ p \in \mathbb{R}_3[x] : p(1) = p(3) = 0 \}.$

$$S_1 \cap T_1 = \{ p \in \mathbb{R}^3[x] : p(0) = p(1) = p(2) = p(3) = 0 \}$$

$$p = a_3(x-x_1)(x-x_2)(x-x_3)$$

$p(0) = 0 = -a_3 x_1 x_2 x_3$ $\qquad a_3 = 0 \Rightarrow p = 0(x)$

$p(1) = 0$

$p(2) = 0$ $\qquad$ etc

$p(3) = 0.$

Sea $T : \mathbb{R}_2[x] \to \mathbb{R}^3$

$$[T]_{B_1}^{B_2} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$

$$B_1 = \{x, \; x^2 - x, \; x - 1\}$$

$$B_2 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Primero

$$S = \{x \in \mathbb{R}^3 : x_1 - x_2 = 0, \; x_1 + x_3 = 0\}$$

$$= \text{gen}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

Busco $p \in \mathbb{R}_2[x] : T(p) = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$$[T]_{B_1}^{B_2} [p]^{B_1} = [T(p)]^{B_2}$$

$$\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \underset{-3}{\alpha} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \underset{-2}{\beta} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \underset{2}{\gamma} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \end{array}\right) \begin{array}{l} F_2 + F_1 \\ F_3 - F_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -2 \end{array}\right)$$

$$[T(p)]^{B_2} = \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} [p]^{B_1} = \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}$$

$$[p]^{B_1} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 1 & 1 & 0 & -2 \\ 0 & 2 & 1 & 2 \end{array}\right) \quad F_2 - F_1$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & 1 & 2 \end{array}\right) \quad F_3 - 2F_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & 0 \end{array}\right)$$

$c = 0 \qquad b = 1 \qquad a = -3$

$$[p]^{B_1} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \longrightarrow$$

$$p = -3x + 1(x^2 - x)$$

$$p = x^2 - 4x$$

La imagen de S es el subespacio gen$\{x^2 - 4x\}$

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \quad T(x) = Ax$

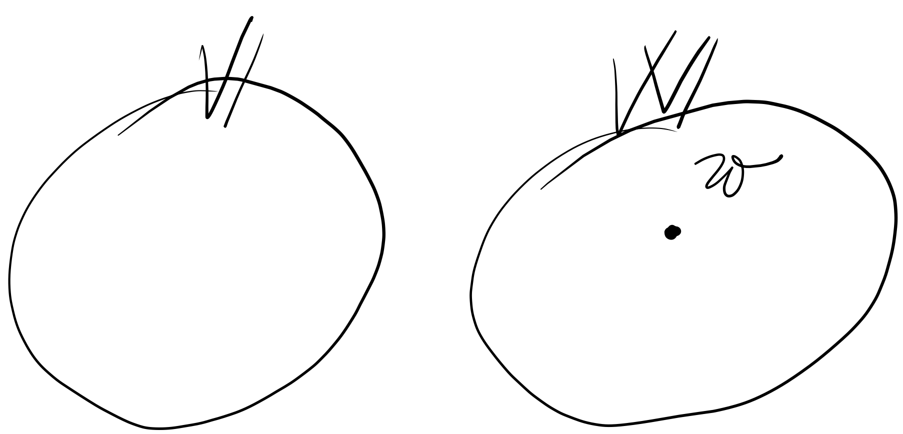
$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad S = \text{gen}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad T^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \nexists$

Hallar $T^{-1}(S) = \{ v \in \mathbb{R}^3 : T(v) \in S \}$

$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ Incompatible

$T^{-1}(S) = \text{Nu}(T)$

$\nexists\, v \in \mathbb{R}^3 : \quad T(v) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$



w un vector

$T^{-1}(w) \begin{cases} \nexists\; w \notin \text{Im}(T) \\ \exists\; w \in \text{Im}(T) \; * \end{cases}$

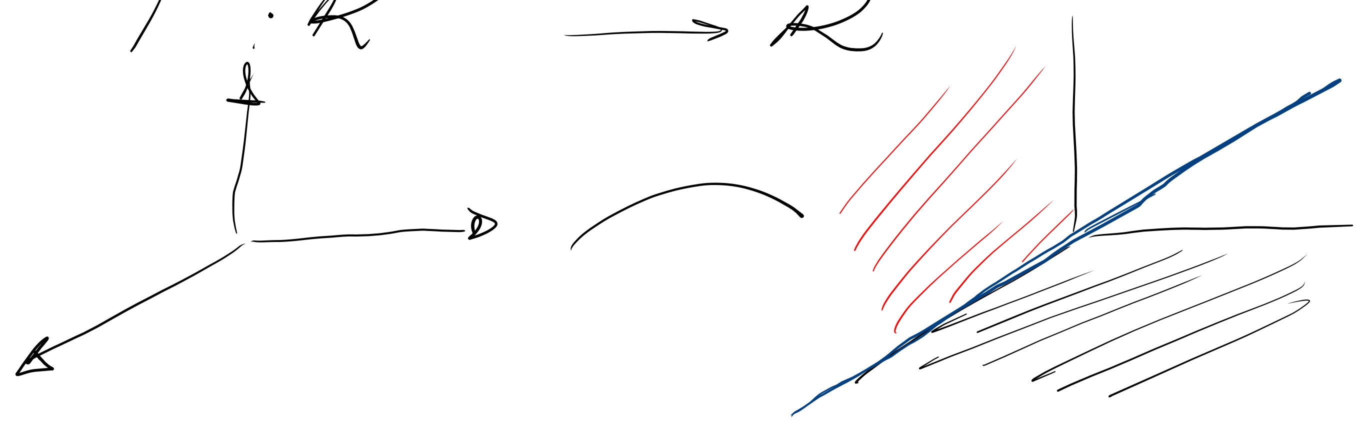
(*) Si T es inyectiva solo un v : $T(v) = w$ sino $v_p + \alpha\, \text{Nu}(T)$

$$T: \mathbb{R}^3 \to \mathbb{R}^3 \qquad T(x) = Ax = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$T^{-1}(S) \qquad S = \text{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \qquad Nu(T) = \text{gen}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$$T^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \nexists \qquad T^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$T: \mathbb{R}^3 \to \mathbb{R}^3$$



$Im(T)$

$$T^{-1}(S) = \text{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$T(x) = Ax$ $\qquad T : \mathbb{R}^n \to \mathbb{R}^m$

$\in \mathbb{R}^{m \times n}$ $\qquad A = [T]_E^E$.

$T(v) = w$ $\qquad T : V \longrightarrow W$

$[T]_{B_1}^{B_2}$ $\qquad B_1 = \{v_1, v_2 \ldots v_n\}$

$B_2 = \{w_1, w_2 \ldots\ldots w_m\}$

cada columna j de $[T]_{B_1}^{B_2}$ es $[T(v_j)]^{B_2}$

$[v]^B = C_{B'}^{B} [v]^{B'}$

Sea $\Pi_{S_1 S_2}$ $\quad S_1 = \{x \in \mathbb{R}^3 : x_1 + x_3 = 0\} = Im(\Pi_{S_1 S_2})$

$S_2 \cap S_1 = \{0\}$ $\quad Nu(\Pi_{S_1 S_2}) = S_2 = gen\left\{\begin{pmatrix}1\\1\\2\end{pmatrix}\right\}$ $\quad S_1 \oplus S_2$

Hallar base de la preimagen de $S = \{x \in \mathbb{R}^3 : x_1 - x_2 = 0\}$

$S \cap S_1 = \{x \in \mathbb{R}^3 : x_1 + x_3 \wedge x_1 - x_2 = 0\} = gen\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix}\right\} \in Im(\Pi)$

$S = gen\left\{\underbrace{\begin{pmatrix}1\\1\\-1\end{pmatrix}}_{\in S_1} \underbrace{\begin{pmatrix}1\\1\\2\end{pmatrix}}_{\in S_2}\right\}$ $\quad \Pi_{S_1 S_2}^{-1}(S) = gen\left\{\begin{pmatrix}1\\1\\-1\end{pmatrix} \begin{pmatrix}1\\1\\2\end{pmatrix}\right\}$

$\Pi_{S_1 S_2}^{-1}\begin{pmatrix}1\\1\\-1\end{pmatrix}$

Otra forma $\Pi_{S_1 S_2}$

$$S_1 = gen\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$S_2 = gen\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \overset{2/3 x_1 - 1/3 x_3}{\alpha} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \overset{-1/3 x_1 + x_2 - \frac{1}{3} x_3}{\beta} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \overset{1/3 (x_1 + x_3)}{\gamma} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{array}{cccc|l} 1 & 0 & 1 & x_1 & \checkmark \\ 0 & 1 & 1 & x_2 & \checkmark \\ -1 & 0 & 2 & x_3 & F_3 + F_1 \end{array} \qquad \begin{array}{cccc} 1 & 0 & 1 & x_1 \\ 0 & 1 & 1 & x_2 \\ 0 & 0 & 3 & x_1 + x_3 \end{array}$$

$$\gamma = 1/3 (x_1 + x_3)$$

$$\beta = x_2 - 1/3 x_1 - 1/3 x_3$$

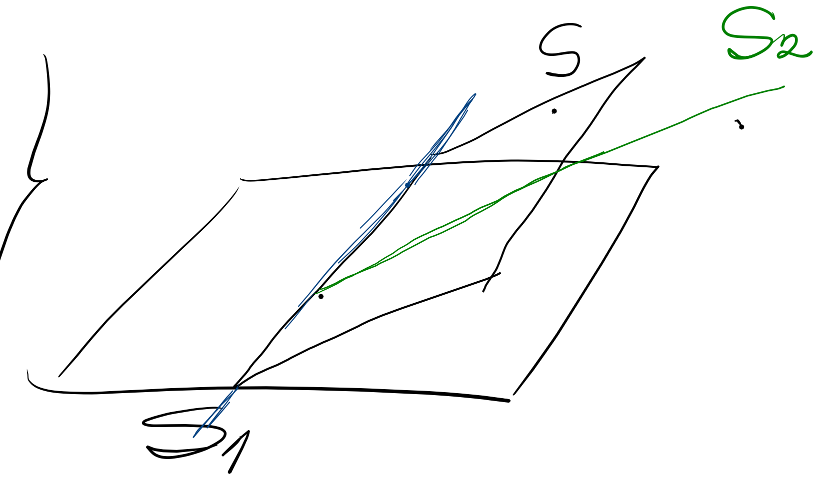
$$\alpha = x_1 - 1/3 x_1 - 1/3 x_3$$

$$\Pi_{S_1, S_2} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (2/3 x_1 - 1/3 x_3) \underbrace{\Pi_{S_1 S_2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} + (-1/3 x_1 + x_2 - 1/3 x_3) \overbrace{\Pi_{S_1 S_2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}^{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} +$$

$$+ (1/3 x_1 + 1/3 x_3) \cancel{\Pi_{S_1 S_2} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}} \rightarrow 0$$

$$\Pi_{S_1S_2}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix} 2/3\,x_1 - 1/3\,x_3 \\ -1/3\,x_1 + x_2 - 1/3\,x_3 \\ -2/3\,x_1 + 1/3\,x_3\end{pmatrix} = \begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}\in S \qquad y_1 = y_2$$

Preimagen de $S = \text{gen}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$



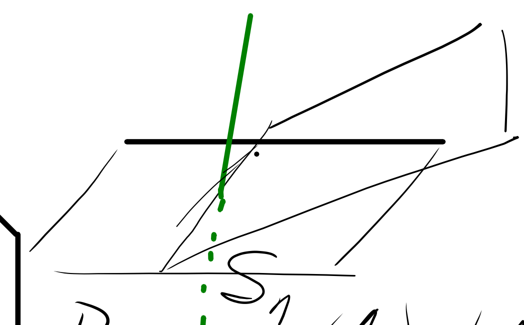
$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} : \Pi_{S_1S_2}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\in S$

$\Rightarrow\ 2/3\,x_1 - 1/3\,x_3 = -1/3\,x_1 + x_2 - 1/3\,x_3$

$x_1 = x_2$

Rta Base $\Pi^{-1}_{(S_1,S_2)}(S) = \left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

$x_1 - 3x_2 + x_3 = 0$



Base $= \left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}1\\1\\2\end{pmatrix}\right\}$

$\in S_1\cap S \qquad \in Nu(\Pi)$

Preimagen de $S = \{v\in\mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$.

$y_1 + y_2 + y_3 = 0$

$\cancel{2/3\,x_1} - 1/3\,x_3 - 1/3\,x_1 + x_2 - \cancel{1/3\,x_3} - \cancel{2/3}\,x_1 + \cancel{1/3\,x_3} = 0$

$\cdot\ A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix}$ pic $\mathbb{R}^4$ $\quad \underline{P_{Col(A)}(x) = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}}$

$d(x, 0) = 2\sqrt{5}$ $\qquad x = \underline{x_{Col(A)}} + \underline{x_{(Col(A))^{\perp}}} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

$v \in (Col(A))^{\perp}$ și $v \perp C_j \quad 1 \leq j \leq 4$

$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} \quad \langle v ; C_1 \rangle = v_1 + v_2 + v_3 + v_4 = 0 \qquad \langle v ; C_4 \rangle = -v_2 + v_4 = 0$

$\langle v ; C_2 \rangle = v_2 + v_4 = 0 \qquad \cancel{\langle v , C_3 \rangle}$

$v_2 = v_4 = 0 \Rightarrow v_1 + v_3 = 0 \Rightarrow (Col(A))^{\perp} = gen\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} \right\}$

$d^2(x, 0) = \|x\|^2 = \langle x, x \rangle = \| x_{Col(A)} \|^2 + \| x_{(Col(A))^{\perp}} \|^2$

Pitágoras

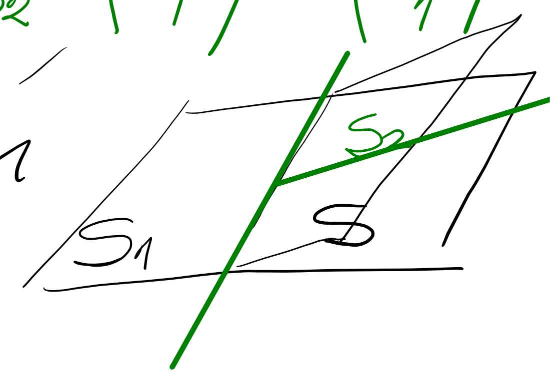
$= 16 + |\alpha|^2 \cdot 2 = 20 \Rightarrow |\alpha|^2 = \frac{20-16}{2} \Rightarrow \alpha = \pm\sqrt{2}$

$\Pi : \mathbb{R}^3 \to \mathbb{R}^3$ $\quad \Pi \circ \Pi = \Pi$ (proyección)

$$\Pi \begin{pmatrix}1\\2\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} \qquad \Pi \begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$x \in \mathbb{R}^3 : x_1 = 0$ $\quad S = \text{gen}\left\{ \begin{pmatrix}0\\1\\0\end{pmatrix}, \underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{\in S_1} \right\}$ $\Rightarrow$ $\Pi_{S_1 S_2} \begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Π proyecta sobre $S_1 = \text{gen}\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$ en la dirección de S_2



$$\begin{pmatrix}1\\2\\1\end{pmatrix} = v_{S_1} + v_{S_2} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\2\end{pmatrix} = u_{S_1} + u_{S_2} = \begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$\Pi_{S_1 S_2} \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\-1\end{pmatrix}$$

$$S_2 = \text{gen}\left\{ \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} = \underbrace{\overset{0}{\alpha} \begin{pmatrix}1\\1\\0\end{pmatrix} + \overset{-1}{\beta} \begin{pmatrix}0\\0\\1\end{pmatrix}}_{\in S_1} + \overbrace{\underset{1}{\gamma} \begin{pmatrix}0\\1\\1\end{pmatrix}}^{\in S_2}$$

$\Rightarrow$ Imagen de S es $S' = \text{gen}\left\{ \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$

$\mathbb{R}^4$ $\quad p = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 3 \end{pmatrix}$ $\quad S = \text{gen}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\}$ $\quad p \in C$

$$d(p, S) = \| P_{S^\perp}(p) \| \qquad S^\perp = \text{gen}\{v\}$$

$$\left\langle v \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle = 0 \qquad \left\langle v \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle = 0 \qquad \left\langle v \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\rangle = 0 .$$

$$v_1 + v_2 = 0 \qquad -v_1 + v_4 = 0 \qquad v_2 + 2v_3 + v_4 = 0$$

$$v_2 = -v_1 \qquad v_4 = v_1 \qquad 2v_3 = 0 \Rightarrow v_3 = 0$$

$$S^\perp = \text{gen}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} \qquad P_{S^\perp} p = \frac{\left\langle p \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}}{\left\| \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\|^2}$$

$$= \frac{6 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}}{3} = 2 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \qquad \Rightarrow d(p, S) = \left\| \begin{pmatrix} 2 \\ -2 \\ 0 \\ 2 \end{pmatrix} \right\| = 2\sqrt{3}$$

$B_1 = \{x + x^2 ; p \quad q\}$ $\quad \mathbb{R}_2[x]$

$[2 - 3x + x^2]^{B_1} = \begin{pmatrix} 4 \\ -1 \\ 1 \end{pmatrix}$ $\qquad [x + 3x^2]^{B_1} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$

Hallar p y q

$$\left.\begin{array}{l} 2 - 3x + x^2 = 4(x + x^2) - p + q \\ x + 3x^2 = 2(x + x^2) + p + q \end{array}\right\} \oplus$$

(a)

$2 - 2x + 4x^2 = 6x + 6x^2 + 2q \Rightarrow q = 1/2\left(2 - 2x + 4x^2 - 6x - 6x^2\right)$

$q = 1 - 4x - x^2 \Rightarrow$ ~~de~~ (a) $\quad p = x + 3x^2 - 2x - 2x^2 - q$

$p = -x + x^2 - 1 + 4x + x^2 = 2x^2 + 3x - 1$.

$\mathbb{R}^{2\times 2}$ $\quad \langle A B\rangle = tr(B^T A)$

$$S = gen\left\{ \underbrace{\begin{pmatrix}1&1\\0&0\end{pmatrix}}_{A_1} \underbrace{\begin{pmatrix}1&0\\1&0\end{pmatrix}}_{A_2} \underbrace{\begin{pmatrix}0&0\\1&1\end{pmatrix}}_{A_3} \right\} \qquad d(A,S)$$

$$A = \begin{pmatrix}1&0\\0&0\end{pmatrix} \qquad S^{\perp} = gen\left\{ \overbrace{\begin{pmatrix}1&-1\\-1&1\end{pmatrix}}^{B} \right\}$$

$$d(A,S) = \| P_{S^{\perp}}(A) \| \qquad S^{\perp} = gen\left\{ \begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix} \right\}$$

$$\left\langle \begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix} \; A_j \right\rangle = 0 \qquad \begin{cases} a_{11}+a_{12}=0 \\ a_{11}+a_{21}=0 \\ a_{21}+a_{22}=0 \end{cases} \qquad \begin{matrix} a_{12}=a_{21} \\ a_{11}=a_{22} \end{matrix}$$

$$a_{22} = -a_{21}$$

$$P(A,S^{\perp}) = \frac{\langle A,B\rangle \; B}{\|B\|^2} =$$

$$= \frac{1 \begin{pmatrix}1&-1\\-1&1\end{pmatrix}}{4} = \frac{1}{4}\begin{pmatrix}1&-1\\-1&1\end{pmatrix} \Rightarrow d(A,S) = \left\| \tfrac{1}{4}\begin{pmatrix}1&-1\\-1&1\end{pmatrix} \right\|$$

$$= 1/2$$

$$\langle A, B\rangle = tr(B^T A) =$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= tr \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} =$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$= tr \begin{pmatrix} b_{11}a_{11} + b_{21}a_{21} & \dots \\ \dots & b_{12}a_{12} + b_{22}a_{22} \end{pmatrix} = b_{11}a_{11} + b_{12}a_{12} + b_{21}a_{21} +$$

$$+ b_{22}a_{22}$$

3. Sea $\Pi : \mathbb{R}^3 \to \mathbb{R}^3$, la proyección sobre $Col(A)$ en la dirección de $\mathrm{Nul}(A)$ y $\Sigma : \mathbb{R}^3 \to \mathbb{R}^3$, la simetría con respecto a $\mathrm{Nul}(A^T)$ en la dirección de $\mathrm{Fil}(A)$, siendo $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$. Hallar todas las soluciones de la ecuación $\Pi(x) = \Sigma([0 \ \ -3 \ \ 0]^T)$.

$$\mathrm{Fil}(A) = \mathrm{gen}\left\{\begin{pmatrix}1\\-1\\1\end{pmatrix}\begin{pmatrix}2\\1\\2\end{pmatrix}\right\} = \mathrm{gen}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$Col(A) = \mathrm{gen}\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right\} = S_1$$

$$= \mathrm{gen}\left\{\begin{pmatrix}1\\2\\1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\right\} = \mathrm{gen}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$$

$$\Pi_{\underset{S_1}{Col(A)},\ \underset{S_2}{\mathrm{Nul}(A)}}$$

$$Ax = 0$$

$$\begin{cases} x_1 - x_2 + x_3 = 0 \\ 2x_1 + x_2 + 2x_3 = 0 \end{cases} \qquad x_2 = 0$$

$$\mathrm{Nul}(A) = \mathrm{gen}\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}\right\} = S_2$$

$$A^T = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$$

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ -x_1 + x_2 - x_3 = 0 \end{cases} \qquad x_2 = 0$$

$$x_1 + x_3 = 0$$

$$\mathrm{Nul}(A^T) = S_3 = S_2$$

$$\mathrm{Fil}(A) = S_4 = S_1$$

$$\Sigma_{S_2, S_1}$$

Π_{S_1, S_2}

$\Sigma_{S_2 \, S_1}$

$S_1 = \text{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$S_2 = \text{gen}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

todas las soluciones de la ecuación $\Pi(x) = \Sigma([0 \quad -3 \quad 0]^T)$.

$$\Pi_{S_1 S_2}(x) = \Sigma \underbrace{\begin{pmatrix} 0 \\ -3 \\ 0 \end{pmatrix}}_{\in S_1} = \underbrace{\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}}_{\in S_1}$$

$$x = \underbrace{\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}}_{x_p} + \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

2. Sea $T : R_3[x] \to \mathbb{R}^3$ la transformación lineal definida por

$$[T]_{\mathcal{B}_1}^{\mathcal{B}_2} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{bmatrix},$$

donde $\mathcal{B}_1$ y $\mathcal{B}_2$ son las bases definidas por

$$\mathcal{B}_1 = \{x^3 - x + 1,\ x^2 + x,\ -x^2 + x,\ 1\},$$
$$\mathcal{B}_2 = \left\{[1\ \ 1\ \ 1]^T,\ [1\ \ 1\ \ -1]^T,\ [1\ \ -1\ \ 1]^T\right\}.$$

Halle bases de $\mathrm{Im}(T)$ y $\mathrm{Nu}(T)$.

$$[T]_{B_1}^{B_2}[v]^{B_1} = [T(v)]^{B_2}$$

$$\Rightarrow [T]_{B_1}^{B_2}[v]^{B_1} = [0_{\mathbb{R}^3}]^{B_2}$$

$$\in Nu(T)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{matrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \end{matrix}\ \ F_3 - F_1 \quad \begin{matrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{matrix}\ \ F_3 - F_2 \quad \begin{matrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{matrix}$$

$$d = 0 \qquad b = -c \qquad a = -c$$

$$\Rightarrow [v]^{B_1} = \begin{pmatrix} -c \\ -c \\ c \\ 0 \end{pmatrix} \Rightarrow v = -c(x^3 - x + 1) - c(x^2 + x) + c(-x^2 + x)$$

$$v = c(-x^3 - 2x^2 + x - 1)$$

$$Nu(T) = gen\{-x^3 - 2x^2 + x - 1\} \qquad Base\ Nu(T) = \{-x^3 - 2x^2 + x - 1\}$$

$$\underbrace{\dim \operatorname{Nu}(T)}_{1} + \dim \operatorname{Im}(T) = \dim \mathbb{R}_3[x] = 4$$

$$\Rightarrow \dim \operatorname{Im}(T) = 3 \qquad \operatorname{Im}(T) \subseteq \mathbb{R}^3$$

$$\operatorname{Im}(T) = \mathbb{R}^3 \qquad \text{Base}_{\operatorname{Im}(T)} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

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1. Sean $\mathbb{S}_1$ y $\mathbb{S}_2$ los siguientes subespacios de $\mathbb{R}_3[x]$:

$$\mathbb{S}_1 = \{p \in \mathbb{R}_3[x] : p(0) = p(2) = 0\}, \quad \mathbb{S}_2 = \operatorname{gen}\{x^2 - 2x, x^3 + 4x\}.$$

Hallar una base de $\mathbb{R}_3[x]$ que contenga una base de $\mathbb{S}_1$ y una base de $\mathbb{S}_2$.

$\dim S_1 = 2 \qquad \dim S_2 = 2 \qquad S_1 \cap S_2$

$$p \in S_2 \Rightarrow p = \alpha \underbrace{(x^2 - 2x)}_{\in S_1 \cap S_2} + \beta(x^3 + 4x)$$

$$\text{Si } p \in S_1 \Rightarrow p(0) = p(2) = 0$$

$$\beta = 0$$

$$p(0) = 0 \quad \forall \alpha \; \forall \beta \qquad p(2) = \alpha(0) + \beta\, 16 = 0$$

$$\text{Base}_{\mathbb{R}_3[x]} = \{ \overbrace{x^3 - 2x^2, \; x^2 - 2x}^{\in S_1}, \; x^3 + 4x \; ; \; x + 1 \}$$

$x^2 - 2x$ $\in S_1 \cap S_2$; $x^2 - 2x$, $x^3 + 4x$: S_2

$x+1$: $\notin S_1$, $\notin S_2$