We note that herein (since $F_k \to 0$) the resultants N, M and Q are linear functions of the real loads F_i, $(i = 1, \dots, n)$, whereas $\bar{N}$, $\bar{M}$ and $\bar{Q}$ are the resultants due to virtual unit forces $\bar{F}_k = 1$.

Using Eqs. (6.12) and (6.13), we may further differentiate the above relation with respect to F_i, and get

$$\delta_{ik} = \int_L \left\{ \frac{N_i N_k}{EA} + \frac{M_i M_k}{EJ} + f_s \frac{Q_i Q_k}{GA} \right\} \mathrm{d}x \,. \qquad (6.51)$$

where now N_i, M_i and Q_i are the resultants due to unit forces $F_i = 1$. This important relation makes it possible to calculate the influence coefficients δ_{ik} from product integrals of the given kind. A brief compilation of product integrals, covering the most commonly encountered functions, is given in Table 6.2.

With the help of these influence coefficients, we may reformulate Eq. (6.50) to give

$$f_k = \sum_i \delta_{ki} F_i \,. \qquad (6.52)$$

Furthermore, we also may use these coefficients to solve statically indeterminate systems. For a system of degree k, we introduce the k redundants as external forces X_i on the primary structure, and thus get from Eq. (6.52)

$$0 = \delta_{k0} + \sum_i \delta_{ki} X_i \,. \qquad (6.53)$$

where the δ_{k0} describe the deformations at point k of the primary structure under the given load system.

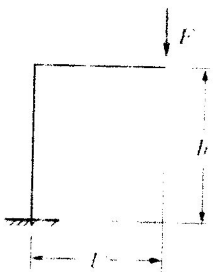
If all redundants X_i are determined from the above system of equations, we finally can describe the distributions of the resultants as, e.g. for the bending moment M,

$$M = M_0 + \sum_i M_i X_i \,. \qquad (6.54)$$

where M_0 is the bending moment of the primary structure under the given load system.

Example 6.1:

The plane frame shown in the figure has a fixed support and carries a vertical load F at the free end. Both members of the frame have constant flexural rigidity EJ.
Determine the horizontal deflection δ_h, the vertical deflection δ_v, and the angle of rotation φ.



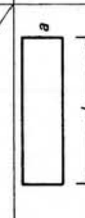
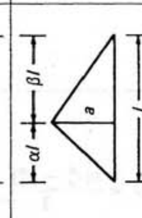
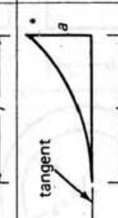
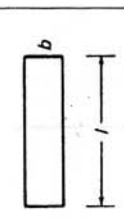
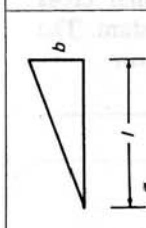
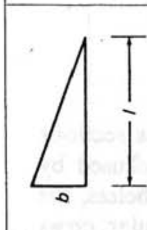
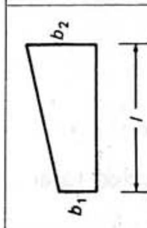
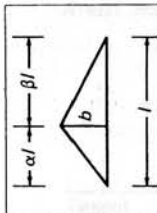
	rectangle k, length ℓ	triangle rising to k, length ℓ	trapezoid k_1–k_2, length ℓ
rectangle i, length ℓ	ℓik	$\frac{1}{2}\ell ik$	$\frac{1}{2}\ell i(k_1 + k_2)$
triangle rising to i, length ℓ	$\frac{1}{2}\ell ik$	$\frac{1}{3}\ell ik$	$\frac{1}{6}\ell i(k_1 + 2k_2)$
triangle falling from i, length ℓ	$\frac{1}{2}\ell ik$	$\frac{1}{6}\ell ik$	$\frac{1}{6}\ell i(2k_1 + k_2)$
trapezoid i_1–i_2, length ℓ	$\frac{1}{2}\ell(i_1 + i_2)k$	$\frac{1}{6}\ell(i_1 + 2i_2)k$	$\frac{1}{6}\ell(2i_1k_1 + i_1k_2 + i_2k_1 + 2i_2k_2)$
quadr. parabola, i, ℓ	$\frac{2}{3}\ell ik$	$\frac{1}{3}\ell ik$	$\frac{1}{3}\ell i(k_1 + k_2)$
quadr. parabola, i, ℓ	$\frac{2}{3}\ell ik$	$\frac{1}{4}\ell ik$	$\frac{1}{12}\ell i(5k_1 + 3k_2)$
quadr. parabola, i, ℓ	$\frac{2}{3}\ell ik$	$\frac{5}{12}\ell ik$	$\frac{1}{12}\ell i(3k_1 + 5k_2)$
quadr. parabola, i, ℓ	$\frac{1}{3}\ell ik$	$\frac{1}{4}\ell ik$	$\frac{1}{12}\ell i(k_1 + 3k_2)$
quadr. parabola, i, ℓ	$\frac{1}{3}\ell ik$	$\frac{1}{12}\ell ik$	$\frac{1}{12}\ell i(3k_1 + k_2)$
triangle, peak i, $\alpha\ell$, $\beta\ell$, ℓ	$\frac{1}{2}\ell ik$	$\frac{1}{6}\ell(1 + \alpha)ik$	$\frac{1}{6}\ell i\{(1 + \beta)k_1 + (1 + \alpha)k_2\}$
$\int_0^\ell [M_k(x)]^2 dx$	ℓk^2	$\frac{1}{3}\ell k^2$	$\frac{1}{3}\ell(k_1^2 + k_1k_2 + k_2^2)$

Table 6.2 Product integrals $\int M_i M_k \,\mathrm{d}x$

APPENDIX H

Values of the Integral $\int M_u M\, dl$

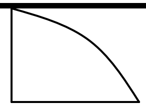
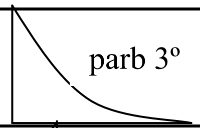
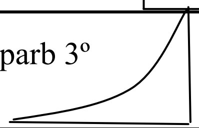
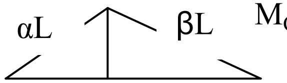
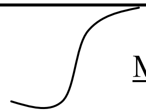
The following table gives the values of the integral $\int^l M_u M\, dl$, needed in the calculation of displacement of framed structures by virtual work (Eq. 7-2). The same tables can be used for the evaluation of the integrals $\int^l N_u N\, dl$, $\int^l V_u V\, dl$, $\int^l T_u T\, dl$, or for the integral over a length l of any two functions which vary in the manner indicated in the diagrams at the top and at the left-hand edge of the table.

M_u \ M				 *	 *	 *
	abl	$\frac{al}{2}(b_1 + b_2)$	$\frac{1}{2}abl$	$\frac{2}{3}abl$	$\frac{1}{3}abl$	$\frac{2}{3}abl$
	$\frac{1}{2}abl$	$\frac{bl}{6}(a_1 + 2a_2)$	$\frac{abl}{6}(1 + \alpha)$	$\frac{1}{3}abl$	$\frac{1}{4}abl$	$\frac{5}{12}abl$
	$\frac{1}{2}abl$	$\frac{bl}{6}(2a_1 + a_2)$	$\frac{abl}{6}(1 + \beta)$	$\frac{1}{3}abl$	$\frac{1}{12}abl$	$\frac{1}{4}abl$
	$\frac{al}{2}(b_1 + b_2)$	$\frac{l}{6}(2a_1b_1 + a_1b_2 + a_2b_1 + 2a_2b_2)$	$\frac{al}{6}[(1 + \beta)b_1 + (1 + \alpha)b_2]$	$\frac{1}{3}al(b_1 + b_2)$	$\frac{al}{12}(b_1 + 3b_2)$	$\frac{al}{12}(3b_1 + 5b_2)$
	$\frac{1}{2}abl$	$\frac{bl}{6}[(1 + \beta)a_1 + (1 + \alpha)a_2]$	$\frac{1}{3}abl$	$\frac{abl}{3}(1 + \alpha\beta)$	$\frac{abl}{12}(1 + \alpha + \alpha^2)$	$\frac{abl}{12}(5 - \beta - \beta^2)$

*2d-degree parabola

EXPRESIONES DE LAS INTEGRALES : $\int_0^L M\underline{M}ds$ EJ = CTE. L = LONGITUD DE BARRA

	$\underline{M}$ $\underline{M}$	$\underline{M}_B$	$\underline{M}_A$ $\underline{M}_B$
M M	$M\underline{M}L$	$1/2\ M\underline{M}L$	$1/2\ M(\underline{M}_A + \underline{M}_B)L$
M_B	$1/2M_B\underline{M}L$	$1/3M_B\underline{M}_BL$	$1/6M_B(\underline{M}_A+2\underline{M}_B)L$
M_A	$1/2M_A\underline{M}L$	$1/6M_A\underline{M}_BL$	$1/6M_A(\underline{M}_B+2\underline{M}_A)L$
M_A M_B	$1/2(M_A+M_B)\underline{M}L$	$1/6\underline{M}_B(M_A+2M_B)L$	$1/6(2M_A\underline{M}_A+M_A\underline{M}_B+M_B\underline{M}_A+2M_B\underline{M}_B)L$
M_C parab. 2º	$2/3M_C\underline{M}L$	$1/3M_C\underline{M}_BL$	$1/3M_C(\underline{M}_A+M_B)L$
$M_B$parab2º	$2/3M_B\underline{M}L$	$5/12M_B\underline{M}_BL$	$1/12M_B(3\underline{M}_A+5\underline{M}_B)L$
M_A parb 2º	$2/3M_A\underline{M}L$	$1/4M_A\underline{M}_BL$	$1/12M_A(5\underline{M}_A+3\underline{M}_B)L$
parb 2º M_B	$1/3M_B\underline{M}L$	$1/4M_B\underline{M}_BL$	$1/12M_B(\underline{M}_A+3\underline{M}_B)L$
M_A parb 2º	$1/3M_A\underline{M}L$	$1/12M_A\underline{M}_BL$	$1/12M_A(3\underline{M}_A+\underline{M}_B)L$
$M_B$parab3º	$3/4M_B\underline{M}L$	$9/20M_B\underline{M}_BL$	$3/20M_B(2\underline{M}_A+3\underline{M}_B)L$

EXPRESIONES DE LAS INTEGRALES : $\int_0^L M\underline{M}ds$	EJ = CTE.		L = LONGITUD DE BARRA
M_A parb 3º 	$3/4 M_A \underline{M} L$	$3/10 M_A \underline{M}_B L$	$3/20 M_A (3\underline{M}_A + 2\underline{M}_B) L$
M_A parb 3º 	$1/4 M_A \underline{M} L$	$1/20 M_A \underline{M}_B L$	$1/20 M_A (4\underline{M}_A + \underline{M}_B) L$
parb 3º M_B 	$1/4 M_B \underline{M} L$	$1/5 M_B \underline{M}_B L$	$1/20 M_B (\underline{M}_A + 4\underline{M}_B) L$
αL βL M_C 	$1/2 M_C \underline{M} L$	$1/6 (1+\alpha) M_C \underline{M}_B L$	$1/6 M_C ((1+\beta)\underline{M}_A + (1+\alpha)\underline{M}_B) L$
 $\underline{M}^2 ds$	$\underline{M}^2 L$	$1/3 \underline{M}^2 L$	$1/3 (\underline{M}^2_A + \underline{M}^2_B + \underline{M}_A \underline{M}_B) L$