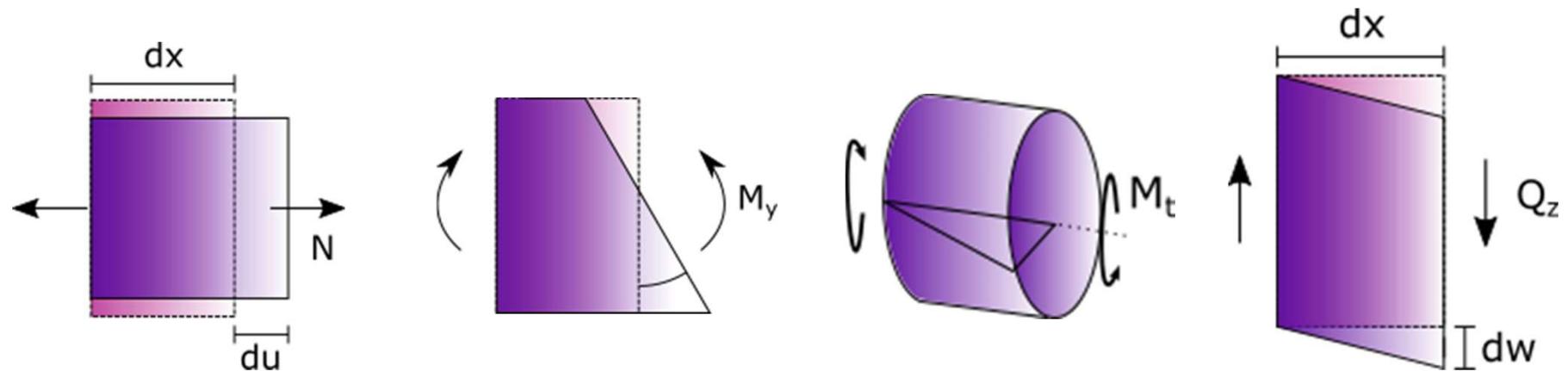




Cálculo de desplazamientos

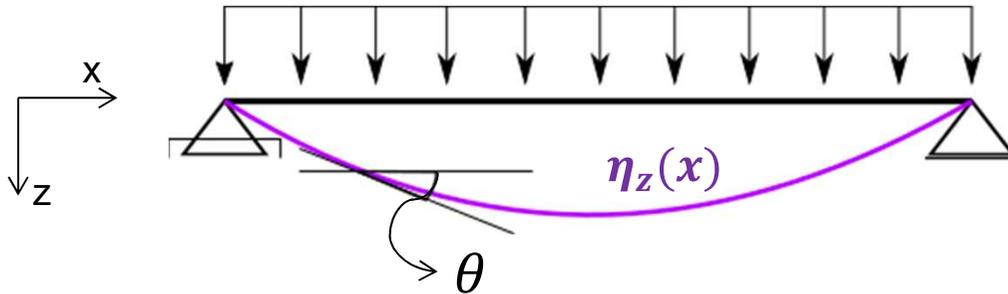


Tania Poletilo - Constanza Ruffinelli - Manuela Medina



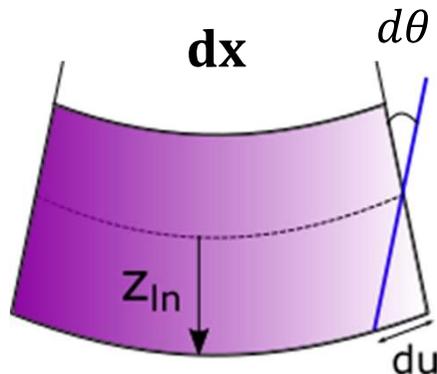
Repaso teórico

Desplazamiento por flexión y corte



$$\theta \approx \text{tg}(\theta) = -\frac{d\eta_z(x)}{dx}$$

Sabemos que $du = \varepsilon_x \cdot dx$



A partir del esquema: $du = d\theta \cdot z_{ln}$

$$d\theta \cdot z_{ln} = \varepsilon_x \cdot dx \longrightarrow \frac{d\theta}{dx} = \chi = \frac{\varepsilon_x}{z_{ln}}$$

Recordando la Ley de Hooke ($\sigma = E \cdot \varepsilon$)
y el cálculo de sigma para EPI ($\sigma = \frac{M_y}{J_y} \cdot z$)

$$\frac{d^2\eta_z(x)}{dx^2} = -\frac{d\theta}{dx} = -\frac{\sigma_x}{E \cdot z_{ln}} = -\frac{M_y \cdot z_{ln}}{J_y \cdot E \cdot z_{ln}} = -\frac{M_y}{J_y \cdot E}$$



Ecuación diferencial de la elástica de deformación

$$\eta_z$$

$$-\theta = \frac{d \eta_z(x)}{dx}$$

$$-\chi = \frac{d^2 \eta_z(x)}{dx^2} = -\frac{M_y(x)}{J_y \cdot E}$$

$$\frac{d^3 \eta_z(x)}{dx^3} = -\frac{1}{J_y \cdot E} \cdot \frac{dM_y(x)}{dx} = -\frac{Q_z(x)}{J_y \cdot E}$$

$$\frac{d^4 \eta_z(x)}{dx^4} = -\frac{1}{J_y \cdot E} \cdot \frac{dQ_z(x)}{dx} = \frac{q_z(x)}{J_y \cdot E}$$



Entonces

Si $q = cte \Rightarrow Q: \text{lineal} \Rightarrow M: \text{cuadrática} \Rightarrow \theta: \text{cúbica}$

$\eta_z(x): \text{polinomio de orden 4} \Rightarrow \eta_z(x) = A \cdot x^4 + B \cdot x^3 + C \cdot x^2 + D \cdot x + F$

$$\frac{d\eta_z(x)}{dx} = -\theta = 4 \cdot A \cdot x^3 + 3 \cdot B \cdot x^2 + 2 \cdot C \cdot x + D$$

$$\frac{d^2\eta_z(x)}{dx^2} = -\chi = -\frac{M_y(x)}{J_y \cdot E} = 12 \cdot A \cdot x^2 + 6 \cdot B \cdot x + 2 \cdot C$$

$$\frac{d^3\eta_z(x)}{dx^3} = -\frac{Q_z(x)}{J_y \cdot E} = 24 \cdot A \cdot x + 6 \cdot B$$

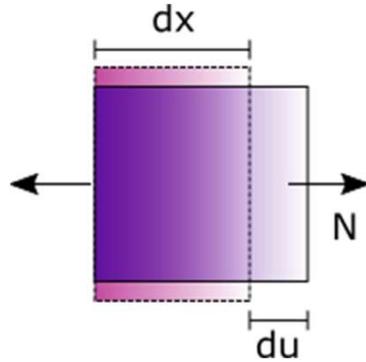
$$\frac{d^4\eta_z(x)}{dx^4} = \frac{q_z}{J_y \cdot E} = 24 \cdot A \Rightarrow A = \frac{q_z}{24 \cdot J_y \cdot E}$$

Como tengo que determinar 4 constantes (B, C, D, F), necesito 4 condiciones de borde



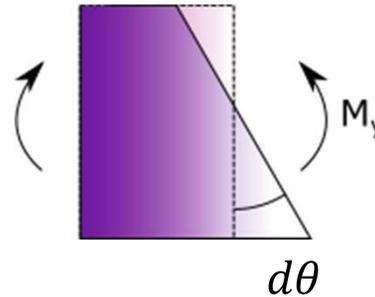
Relaciones entre sollicitaciones y deformada

Axil



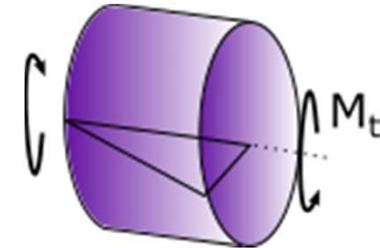
$$du = \varepsilon_x \cdot dx = \frac{N}{E \cdot A} dx$$

Flexión



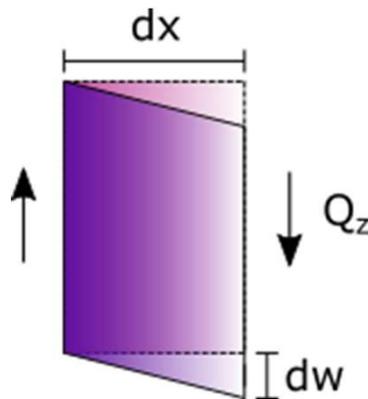
$$-d\theta_y = \chi_y dx = \frac{M_y}{J_y \cdot E} dx$$

Torsión



$$d\theta_x = \chi_x dx = \frac{M_t}{J_p \cdot G} dx$$

Corte



$$dw = K_y \cdot \frac{Q_z}{G \cdot A} dx \quad K_y = \int_A \frac{S^2}{i_y^4 \cdot A \cdot b^2} dA$$

Con K_y :
factor de
forma

Para barras donde la $L \gg a$ (L : longitud de la barra, a : mayor dimensión de la sección), se desprecia la deformación por corte



Cálculo de desplazamientos por TTV

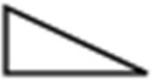
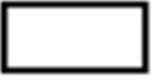
$$W_E = W_I$$

$$(+1) \cdot \eta = \int_L N_{se} du_{dv} + \int_L M_{y_{se}} d\theta_{y_{dv}} + \int_L M_{t_{se}} d\theta_{x_{dv}} + \int_L Q_{se} dw_{dv}$$

$$(+1) \cdot \eta = \int_L N^{se} \cdot \frac{N^{dv}}{E \cdot A} dx + \int_L M_y^{se} \cdot \frac{M_y^{dv}}{J_y \cdot E} dx + \int_L M_t^{se} \cdot \frac{M_t^{dv}}{J_p \cdot G} dx$$

$$\int i(x) \cdot j(x) \cdot dx = \alpha \cdot i \cdot j \cdot L$$

i y j valores máximos de la función

α				
	1	1/2	1/2	2/3
	1/2	1/3	1/6	1/3

Allí la pendiente es cero

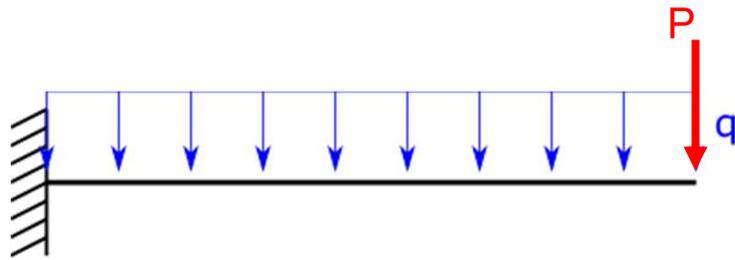
Aclaración

Ésta es una tabla simple, hay otras con más combinaciones pero siempre se pueden descomponer los diagramas en una suma de éstas

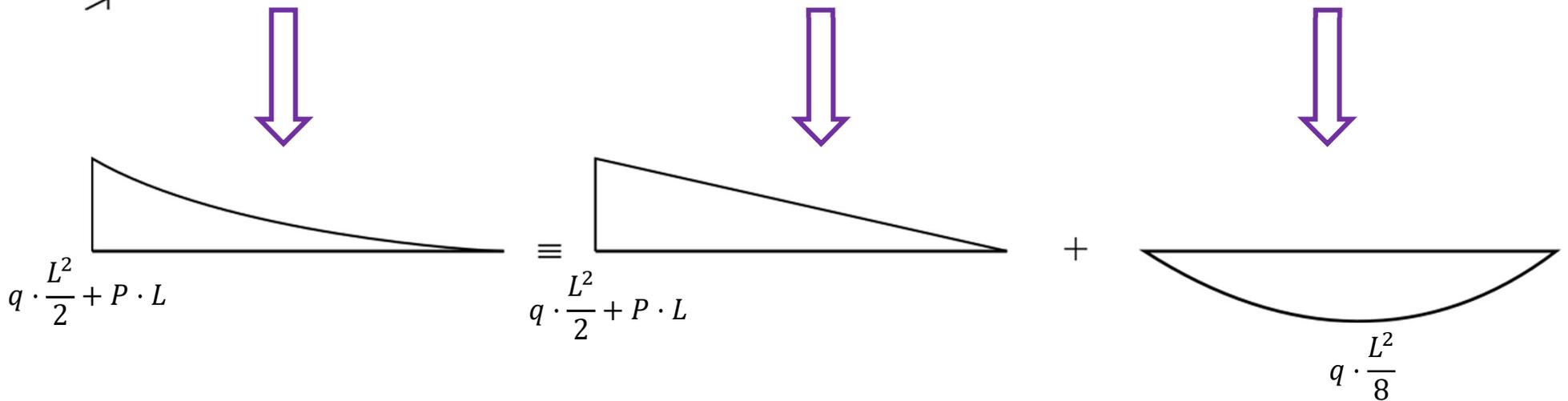
<https://campus.fi.uba.ar/mod/folder/view.php?id=43794>



Descomposición de diagramas

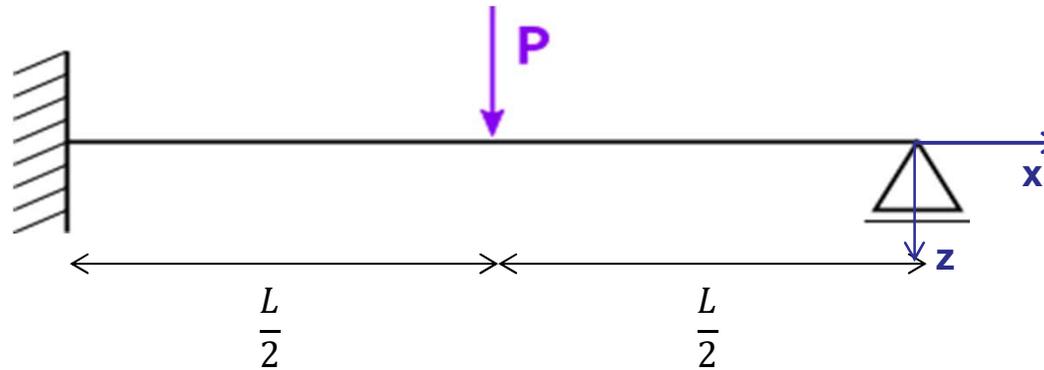


Cómo ésta función no la encuentro en tabla debo dividirla en suma de funciones que si estén





Resolución de la elástica por ecuación diferencial y condiciones de borde



¿Cuántas funciones necesito?

¿Cuántas condiciones de borde necesito?

Cada función $w_i(x)$ tiene 4 incógnitas

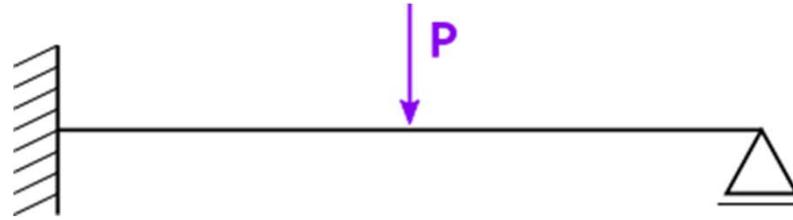
$$w(x) = \begin{cases} w_1(x) & \text{si } 0 < x < \frac{L}{2} \\ w_2(x) & \text{si } \frac{L}{2} < x < L \end{cases}$$



8 condiciones de vínculo



Condiciones de borde



$$1) w_1(x = 0) = 0$$

$$2) \theta_1(x = 0) = 0$$

$$3) w_2(x = L) = 0$$

$$4) M(x = L) = 0 \rightarrow \chi_2(x = L) = 0$$

$$5) w_1\left(x = \frac{L}{2}\right) = w_2\left(x = \frac{L}{2}\right)$$

$$6) \theta_1\left(x = \frac{L}{2}\right) = \theta_2\left(x = \frac{L}{2}\right)$$

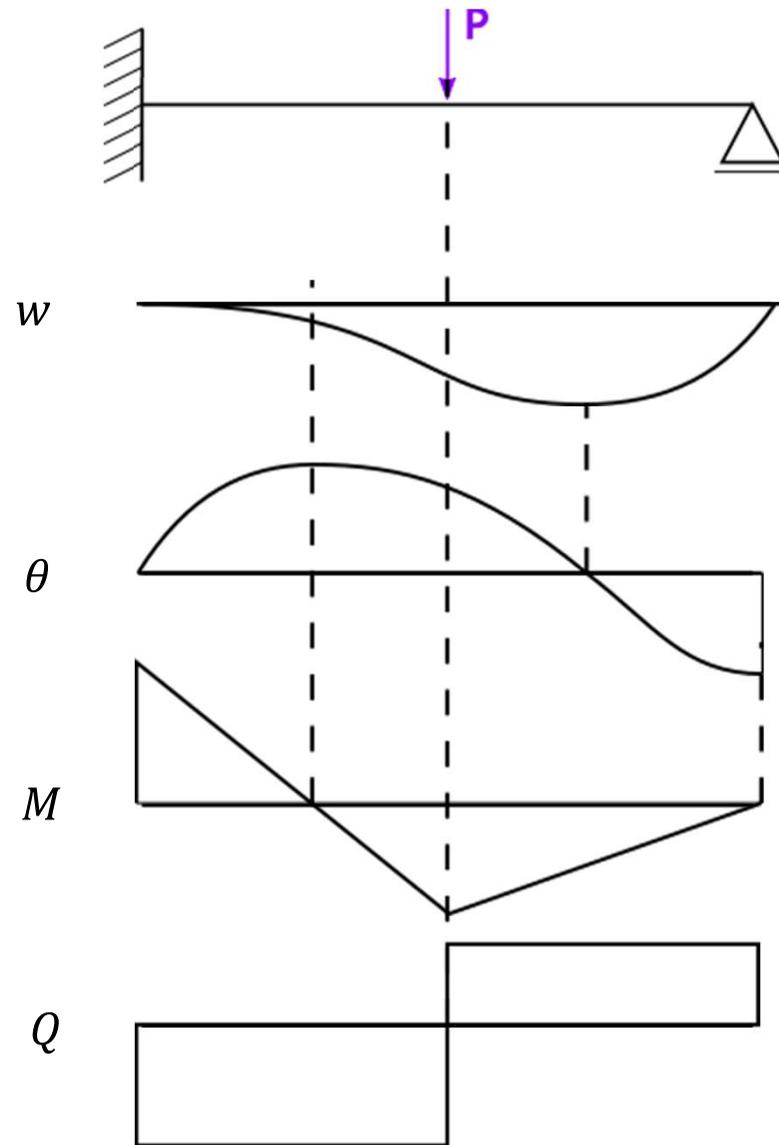
$$7) M_1\left(x = \frac{L}{2}\right) = M_2\left(x = \frac{L}{2}\right)$$

$$8) Q_1\left(x = \frac{L}{2}\right) = Q_2\left(x = \frac{L}{2}\right) + P$$

Una vez determinadas las 8 condiciones de borde tenemos un sistema de 8 ecuaciones y 8 incógnitas.

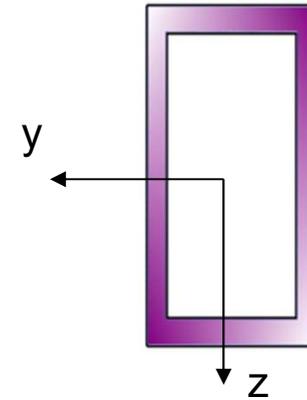
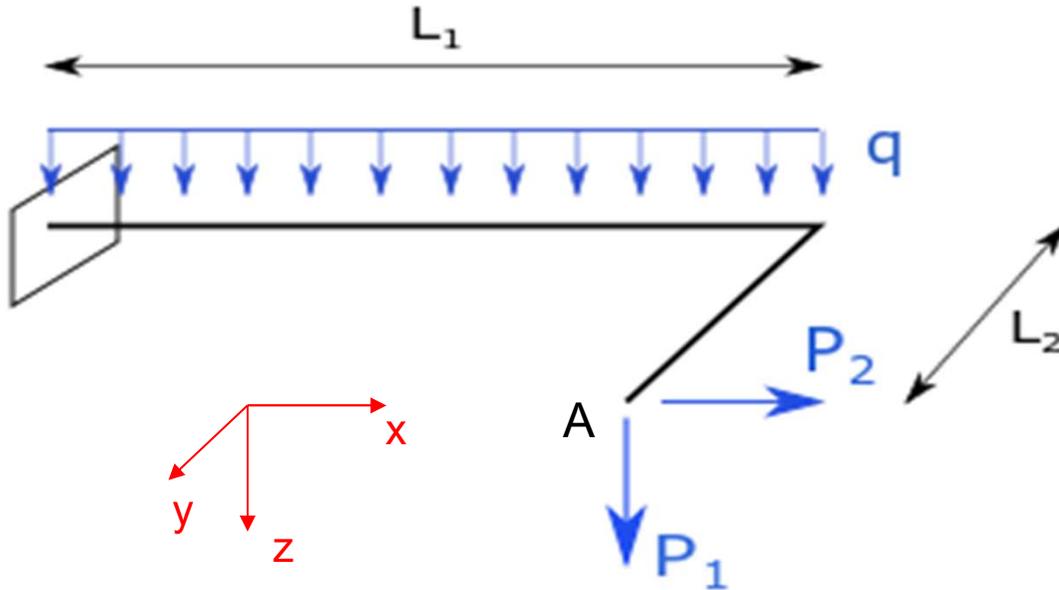


Diagramas





Ejercicio 2: Calcular el desplazamiento vertical en A y θ_{Ax} .



Perfil: 100 x 250 x 10 mm

$$J_y = 4515 \text{ cm}^4$$

$$J_z = 1040 \text{ cm}^4$$

$$J_t = 2777 \text{ cm}^4$$

$$E = 21000 \frac{\text{kN}}{\text{cm}^2}$$

$$G = 8000 \frac{\text{kN}}{\text{cm}^2}$$

Cargas: $P_1 = 5 \text{ kN}$ $L_1 = 4 \text{ m}$

$P_2 = 10 \text{ kN}$ $L_2 = 1 \text{ m}$

$$q = 5 \frac{\text{kN}}{\text{m}}$$



Resolución:

Calculamos cada desplazamiento con TTV

$$W_E = W_i$$

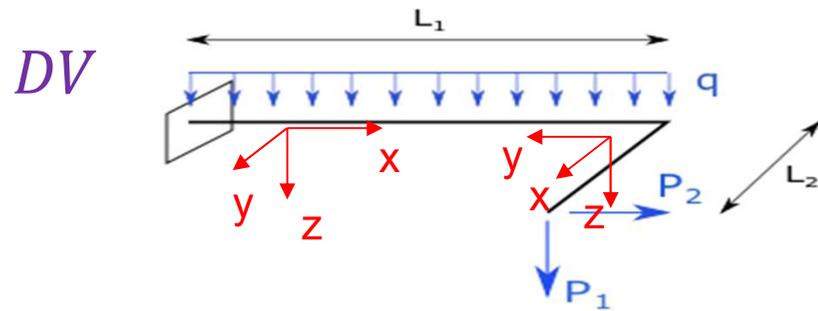
$$(+1) \cdot \eta = \int_L N^{se} \cdot \frac{N^{dv}}{E \cdot A} dx + \int_L M_{en}^{se} \cdot \frac{M_{en}^{dv}}{J_{en} \cdot E} dx + \int_L M_t^{se} \cdot \frac{M_t^{dv}}{J_t \cdot G} dx$$

Por lo que necesitamos:

- Diagramas del DV
- Diagramas de cada SE



Diagramas de características



$$P_1 = 5 \text{ kN}$$

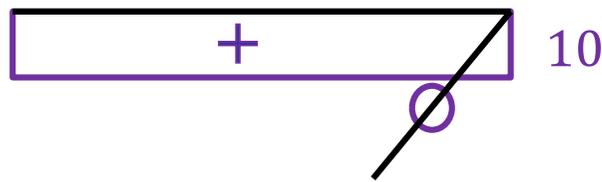
$$L_1 = 4 \text{ m}$$

$$P_2 = 10 \text{ kN}$$

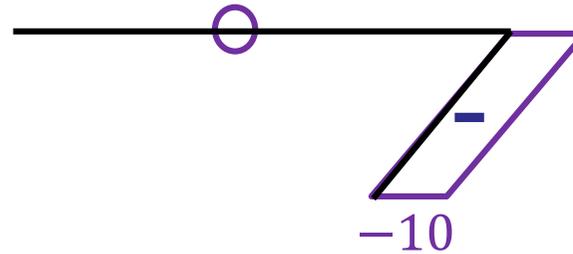
$$L_2 = 1 \text{ m}$$

$$q = 5 \frac{\text{kN}}{\text{m}}$$

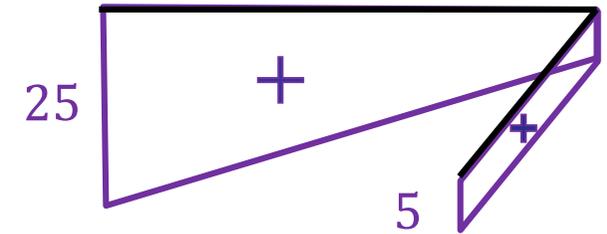
$N[\text{kN}]$



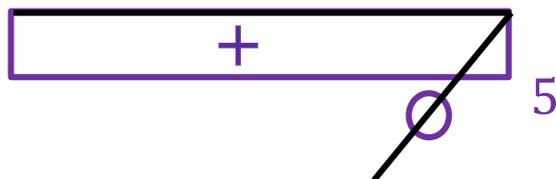
$Q_y[\text{kN}]$



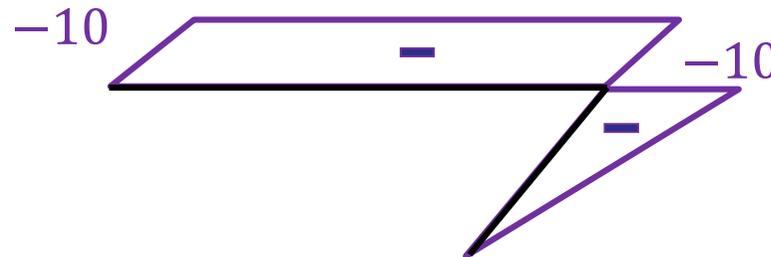
$Q_z[\text{kN}]$



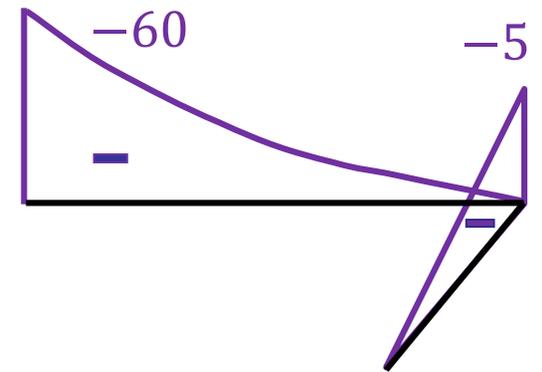
$M_t[\text{kN m}]$



$M_z[\text{kN m}]$

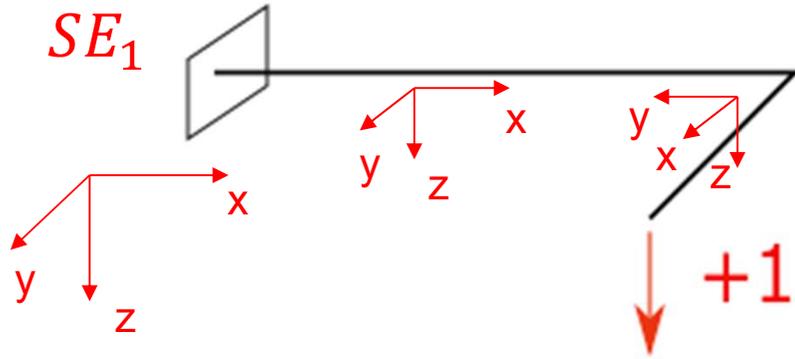


$M_y[\text{kN m}]$

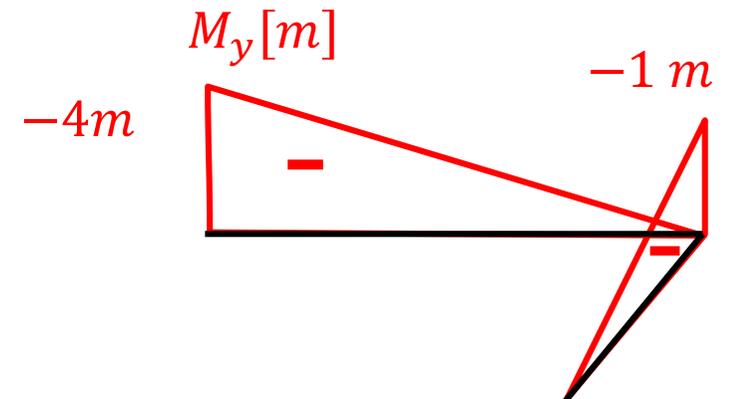
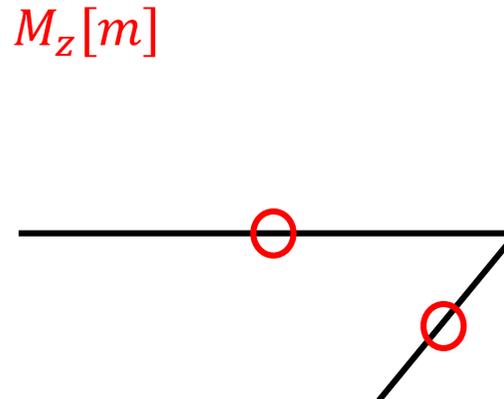
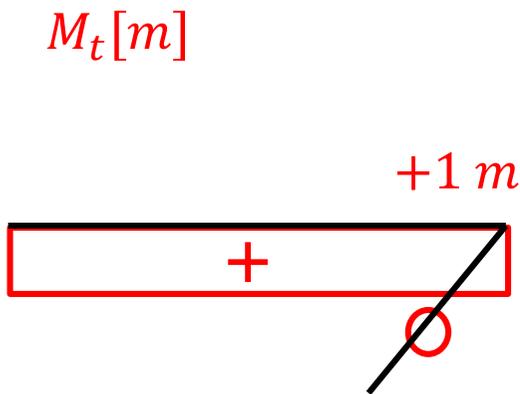
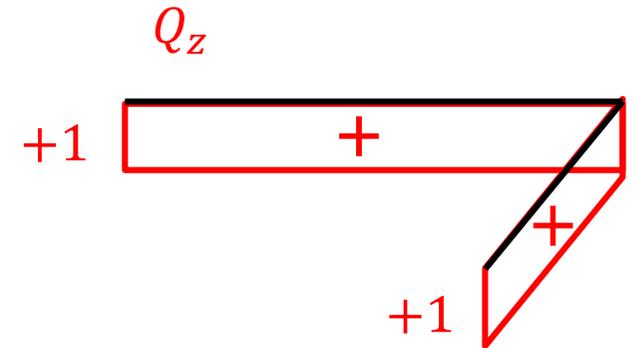
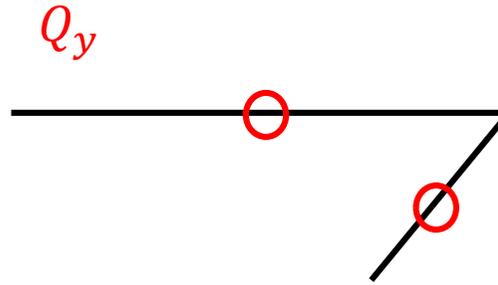
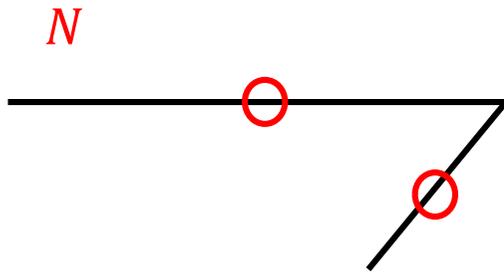




Desplazamiento vertical: η_A

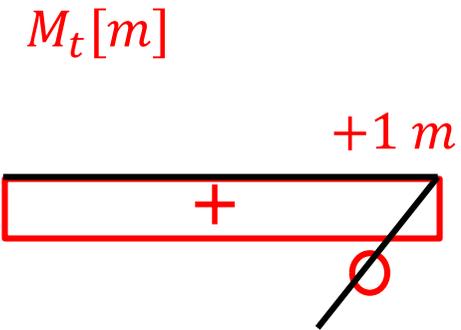
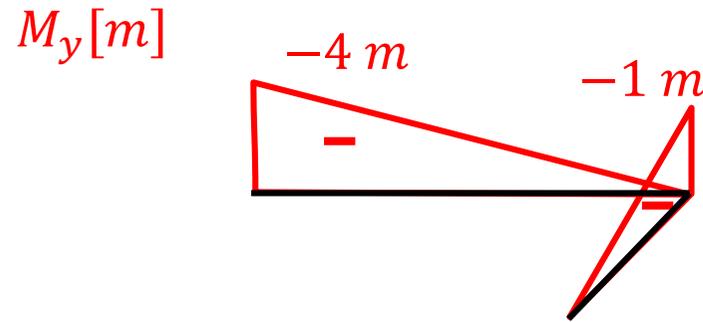


Quiero un desplazamiento vertical, por lo que en el SE debo poner una fuerza unitaria vertical en A

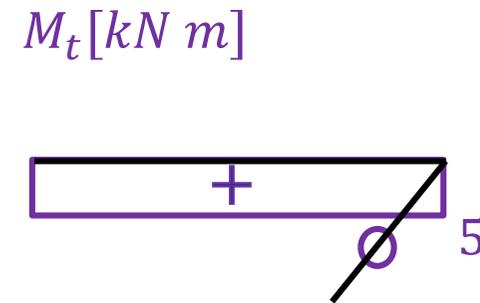
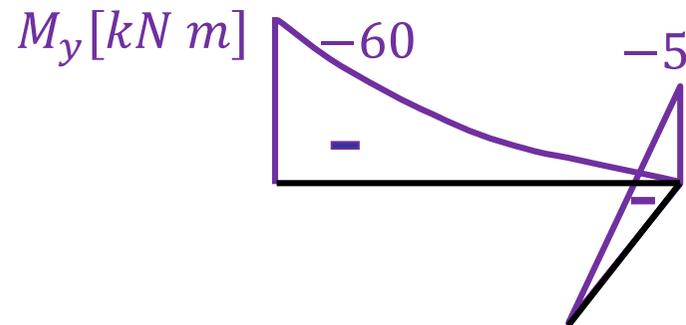




SE_1



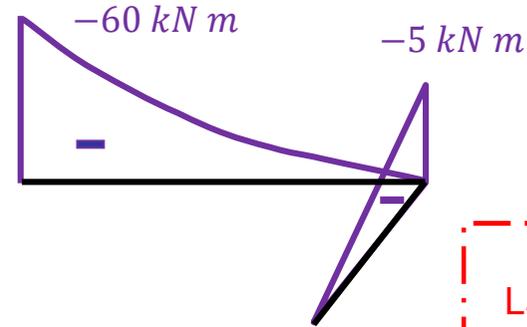
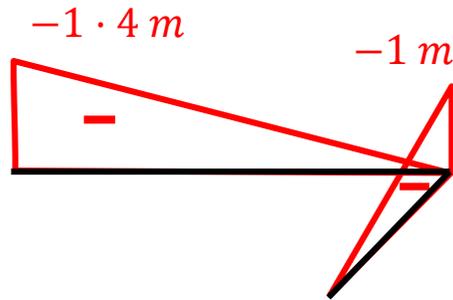
DV



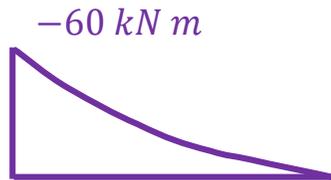
$$\eta_A = \int_L M_{se} \cdot \frac{M_{dv}}{E \cdot J} dx + \int_L Mt_{se} \cdot \frac{Mt_{dv}}{G \cdot J_t} dx = \eta_A^{M_y} + \eta_A^{M_t}$$



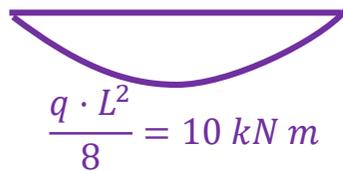
$\eta_A^{M_y}$



Cuidado!
La tangente no es nula,
por lo tanto el coeficiente
no esta en tabla.



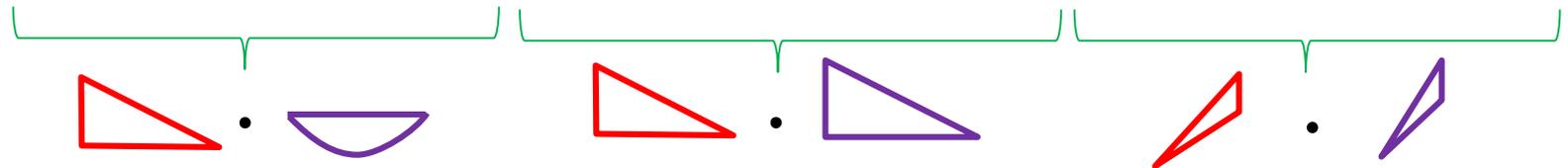
=



+



$$\int_L M_{se} \cdot \frac{M_{dv}}{E \cdot J} dx = \frac{1}{3} \cdot (-4m) \cdot \frac{10kNm}{E \cdot J_y} \cdot 4m + \frac{1}{3} \cdot (-4m) \cdot \frac{(-60kNm)}{E \cdot J_y} \cdot 4m + \frac{1}{3} \cdot (-1m) \cdot \frac{(-5kNm)}{E \cdot J_y} \cdot 1m$$



$$\eta_A^{M_y} = -0,562 \text{ cm} + 3,375 \text{ cm} + 0,018 \text{ cm}$$

$$\eta_A^{M_y} = 2,831 \text{ cm}$$



$$\int_L M_{t_{se}} \cdot \frac{M_{t_{dv}}}{G \cdot J_t} dx = 1 \cdot (+1 m) \cdot \frac{(+5 kN m)}{G \cdot J_t} \cdot 4 m$$

$$\eta_A^{M_t} = +0,900 cm$$

Desplazamiento total:

$$\eta_A = \eta_A^{M_y} + \eta_A^{M_t}$$

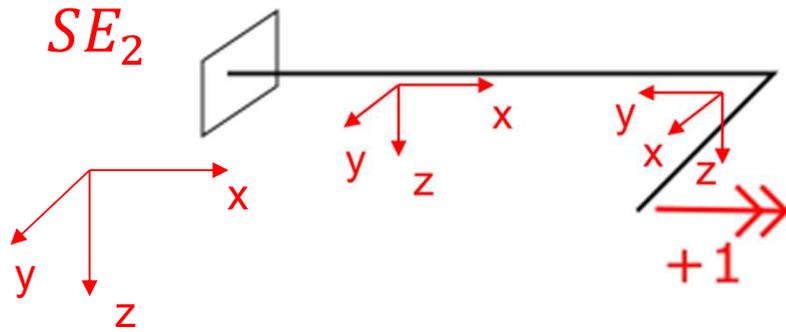
$$\eta_A = 2,831 cm + 0,900 cm$$

$$\eta_A = 3,731 cm$$

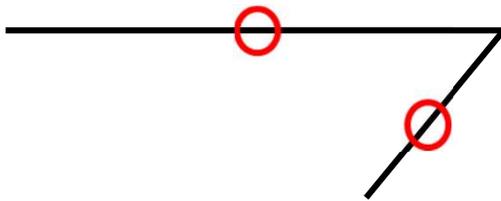


Giro en A: θ_{xA}

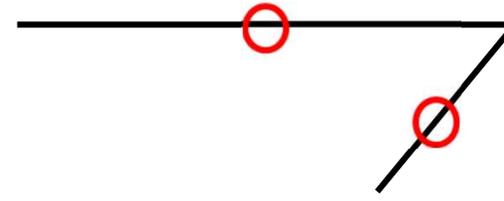
Quiero un giro, por lo que en el SE debo poner un momento unitario en A



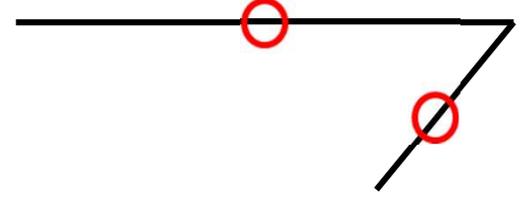
N



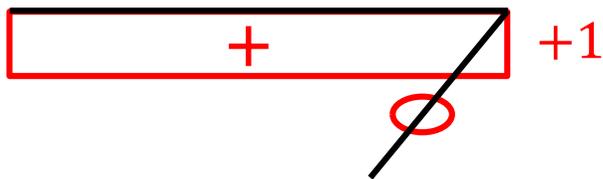
Q_y



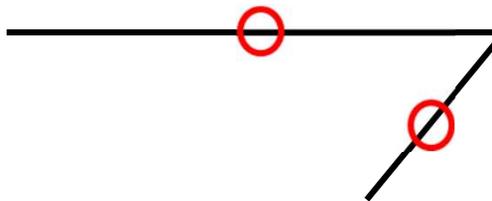
Q_z



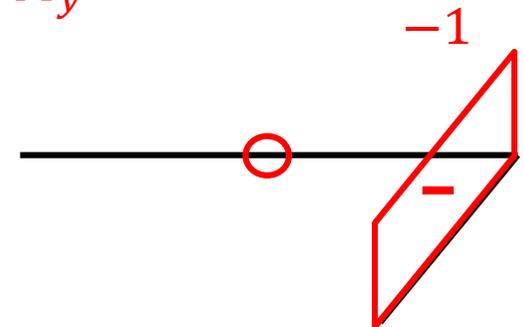
M_t

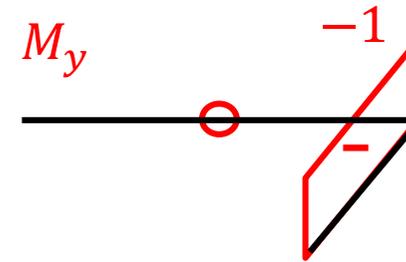
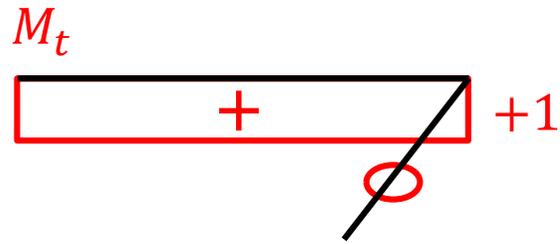
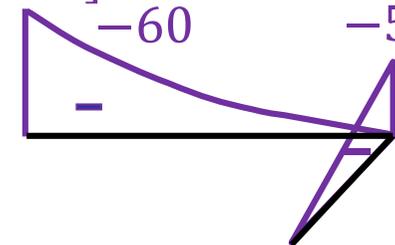
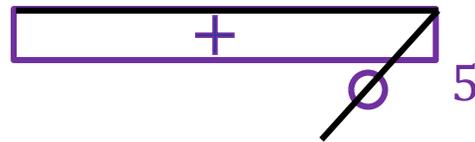


M_z



M_y




 SE_1

 $M_t [kN m]$
 $M_y [kN m]$
 DV


$$\theta_{xA} = \int_L M_{t_{se}} \cdot \frac{M_{t_{dv}}}{G \cdot J_t} dx + \int_L M_{se} \cdot \frac{M_{dv}}{E \cdot J} dx$$

$$\theta_{xA} = 1 \cdot (+1) \cdot \frac{(+5 \text{ kN m})}{G \cdot J_t} \cdot 4 \text{ m} + \frac{1}{2} \cdot (-1) \cdot \frac{(-5 \text{ kN m})}{E \cdot J_y} \cdot 1 \text{ m}$$

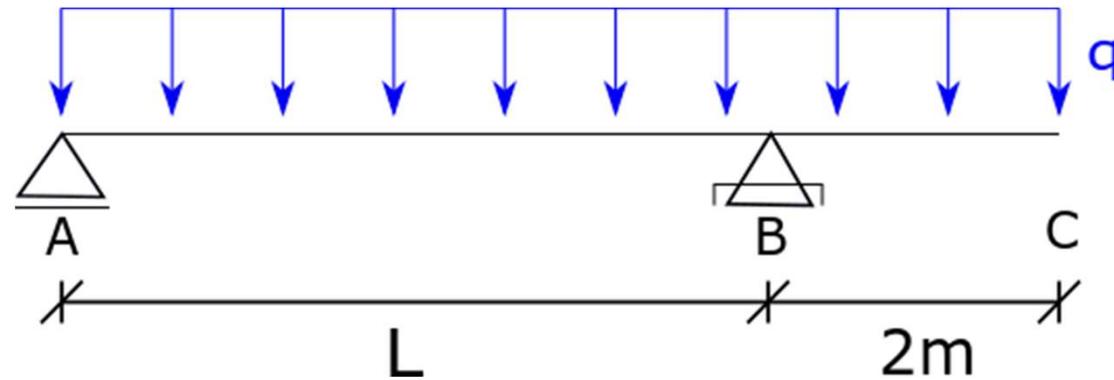
$$\theta_{xA} = 9,00 \cdot 10^{-3} \text{ rad} + 2,63 \cdot 10^{-4} \text{ rad}$$

$$\theta_{xA} = 9,263 \cdot 10^{-3} \text{ rad}$$



Ejercicio 3:

- Calcular la longitud "L" tal que el giro en la sección "B" sea nulo
- Calcular el desplazamiento en "C"



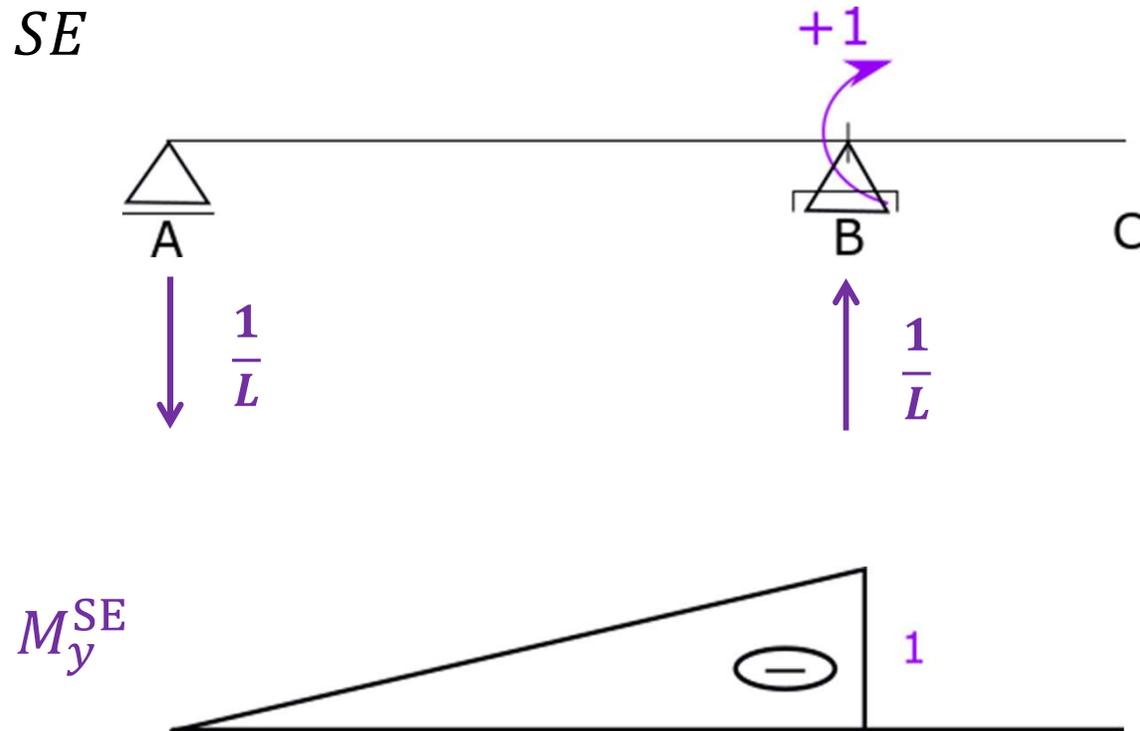
Datos: $E \cdot J = cte$

$$q = 10 \frac{kN}{m}$$



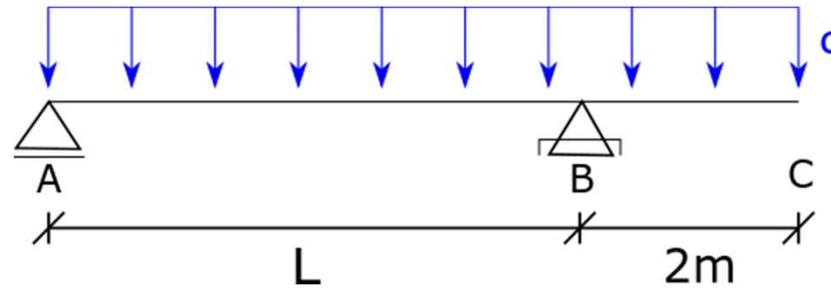
Calculo el giro utilizando TTV

Quiero calcular el giro, por lo tanto en el sistema equilibrado tengo que poner un momento unitario en "B"

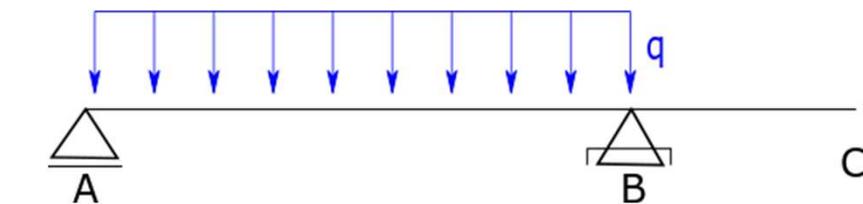
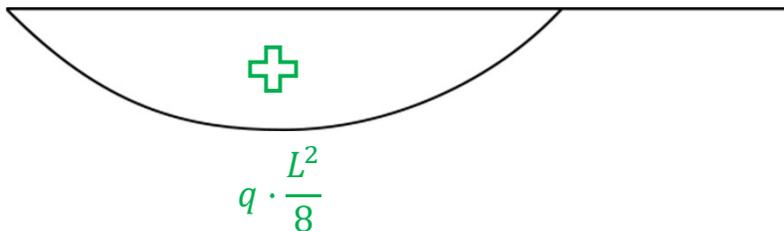
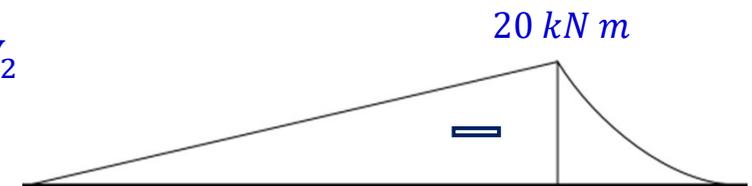


DV 

Opción 1: Realizo los diagramas de características directamente de la estructura completa



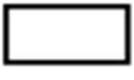
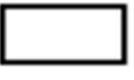
Opción 2: Utilizo superposición de efectos

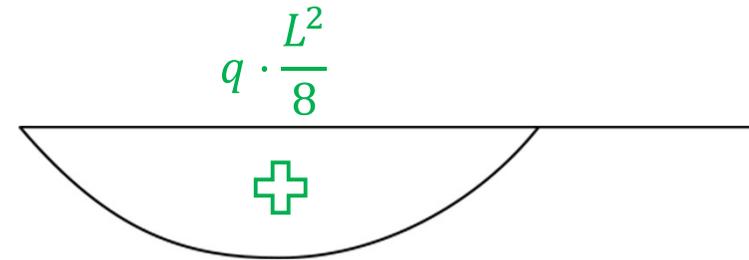
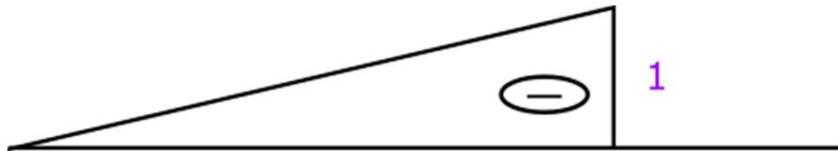
 DV_1  DV_2 



$$\theta_B = \int_L M^{SE} \cdot \frac{M^{DV}}{E \cdot J} dx = \int M^{SE} \cdot \frac{M^{DV_1}}{E \cdot J} dx + \int M^{SE} \cdot \frac{M^{DV_2}}{E \cdot J} dx$$

$$\int M^{SE} \cdot \frac{M^{DV_1}}{E \cdot J} dx \quad \longrightarrow$$

α				
	1	1/2	1/2	2/3
	1/2	1/3	1/6	1/3

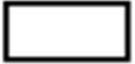
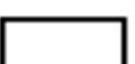


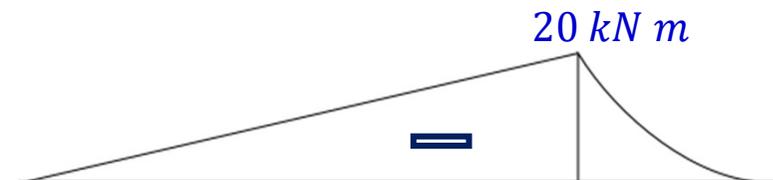
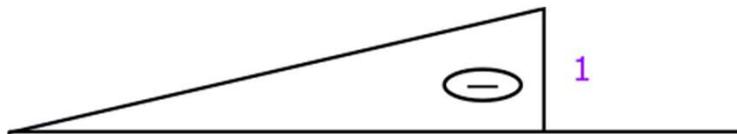
$$\int M^{SE} \cdot \frac{M^{DV_1}}{E \cdot J} dx = \frac{1}{3} \cdot \frac{(-1) \cdot \left(10 \frac{kN}{m} \cdot \frac{L^2}{8}\right) \cdot L}{E \cdot J} = -\frac{5}{12} \frac{kN}{m} \cdot \frac{L^3}{E \cdot J}$$



$$\int M^{SE} \cdot \frac{M^{DV_2}}{E \cdot J} dx$$



α				
	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$
	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$



$$\int M^{SE} \cdot \frac{M^{DV_2}}{E \cdot J} dx = \frac{1}{3} \cdot \frac{(-1) \cdot (-20 \text{ kN m}) \cdot L}{E \cdot J} = \frac{20}{3} \cdot \frac{L}{E \cdot J} \text{ kN m}$$



$$\theta_B = \int M^{SE} \cdot \frac{M^{DV_1}}{E \cdot J} dx + \int M^{SE} \cdot \frac{M^{DV_2}}{E \cdot J} dx$$



Reemplazando y recordando que $\theta_B = 0$

$$0 = -\frac{5}{12} \frac{kN}{m} \cdot \frac{L^3}{E \cdot J} + \frac{20}{3} \cdot \frac{L}{E \cdot J} kN m$$

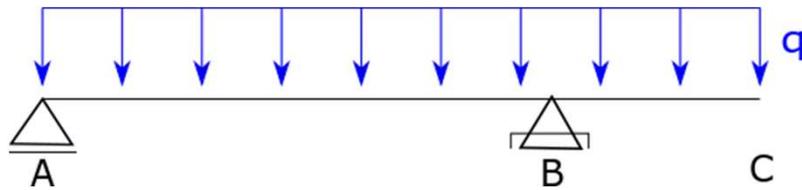
$$\frac{5}{12} \frac{kN}{m} \cdot \frac{L^3}{E \cdot J} = \frac{20}{3} \cdot \frac{L}{E \cdot J} kN m \rightarrow \frac{5}{12} \cdot L^2 = \frac{20}{3} m^2$$

$$L = 4 m$$



b) Desplazamiento en "C"

DV



SE

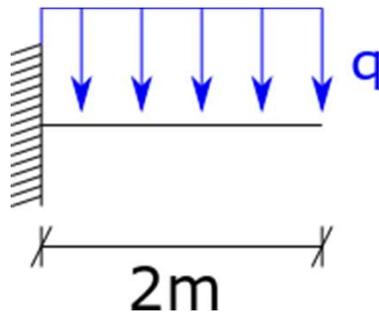


Pero sabemos que en el punto B no hay giros ni desplazamientos



El punto B funciona como un empotramiento

DV



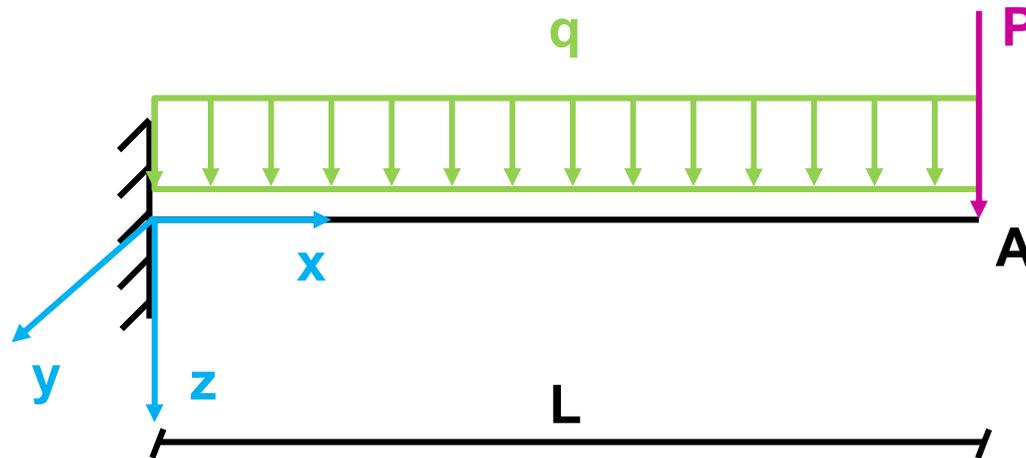
SE



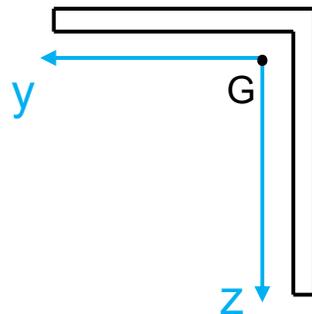


Ejercicio 4:

a) Calcular el desplazamiento máximo del punto A



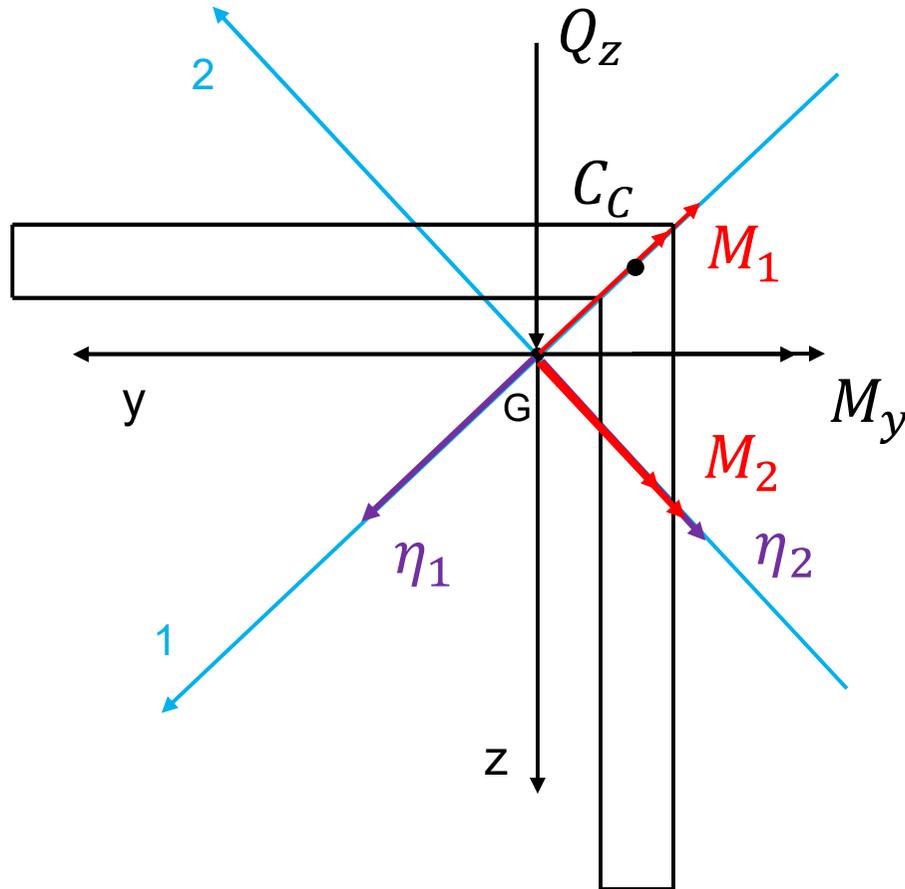
Datos

 L P, q E

Perfil angulo de alas iguales



- Analizamos la sección



¡Observación!
Tenemos que trabajar con los ejes principales de inercia

$$M_1 = M_y \cdot \cos 45^\circ = M_2$$

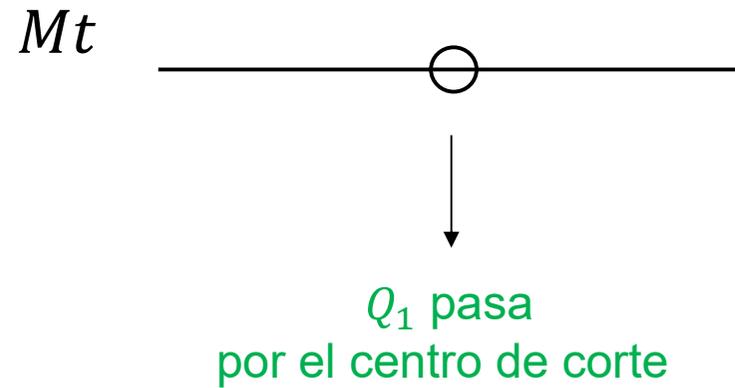
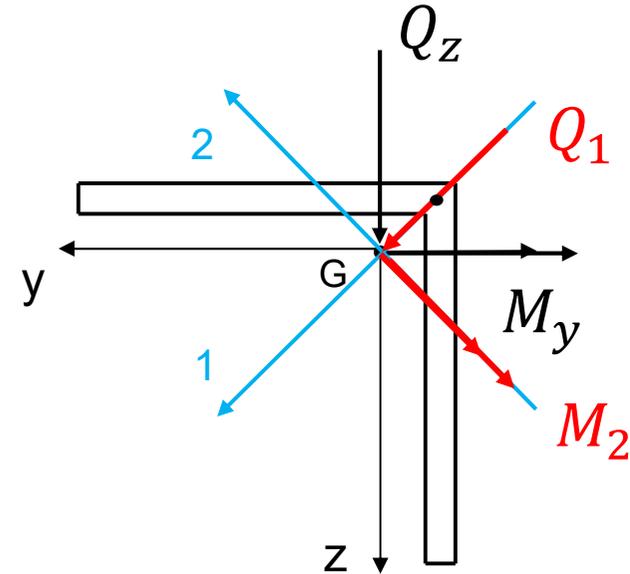
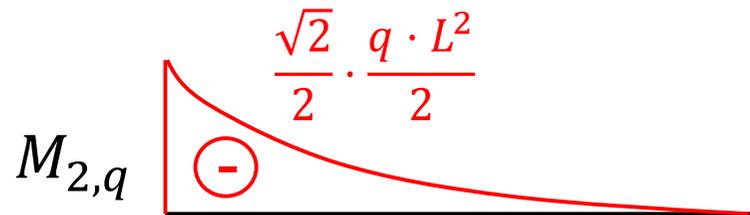
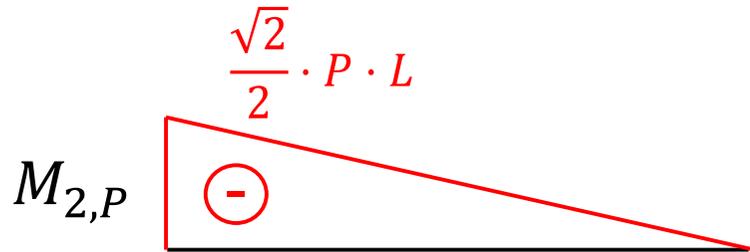
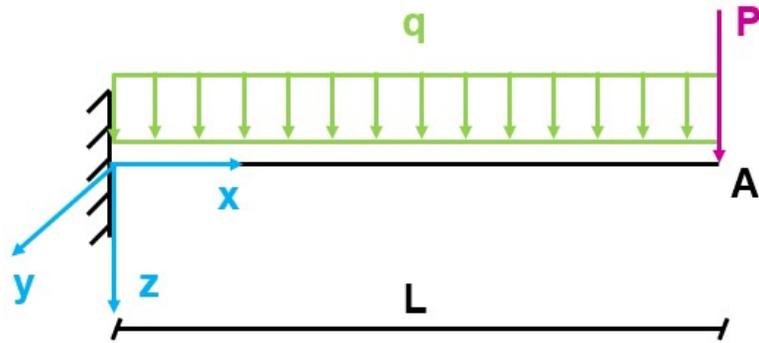
¡Observación!
Aparece un efecto de torsión debido a que Q no pasa por el centro de corte

M_t



Desplazamiento $\eta_{A,1}$

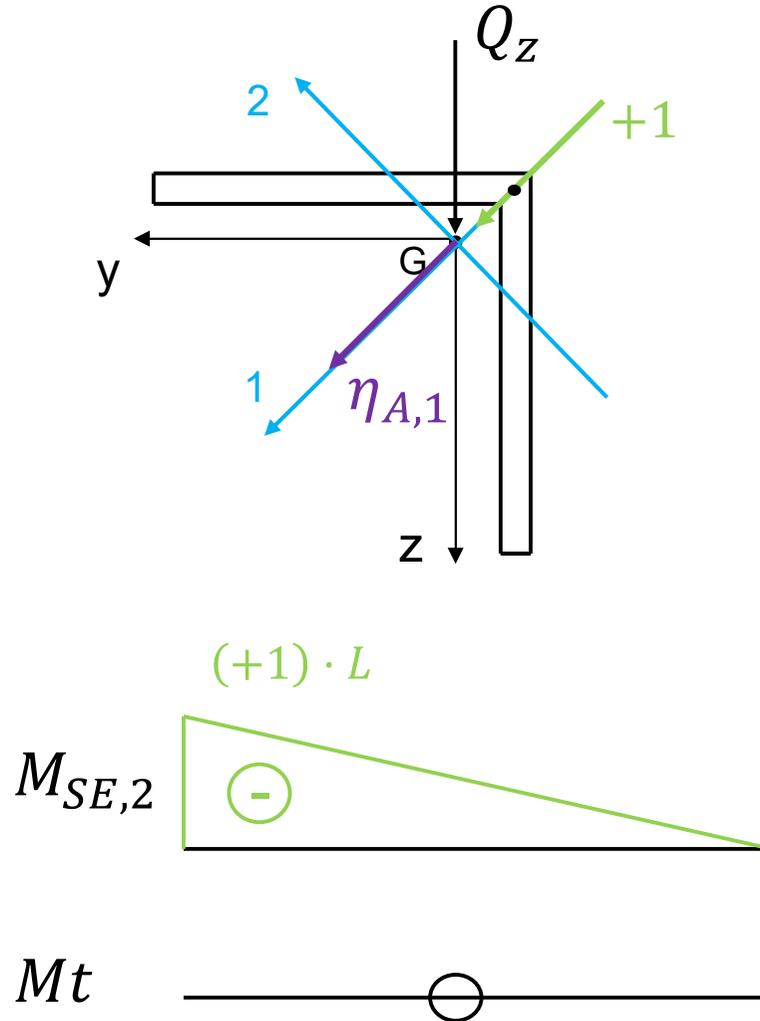
- Diagramas de la deformación virtual





- Sistema equilibrado \longrightarrow Aplicamos fuerza unitaria en **A** con dirección 1

- Sección A





- Aplicamos el **Teorema de los trabajos virtuales**

$$(+1) \cdot \eta_{A,1} = \int \frac{M_2^{SE} \cdot M_2^{DV}}{E \cdot J_2} dx + \underbrace{\int \frac{M_t^{SE} \cdot M_t^{DV}}{G \cdot J} dx}_{=0}$$

$$(+1) \cdot \eta_{A,1} = \int_0^L \frac{\begin{array}{c} (+1) \cdot L \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \cdot \begin{array}{c} M_{2,p} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}}{E \cdot J_2} dx + \int_0^L \frac{\begin{array}{c} (+1) \cdot L \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \cdot \begin{array}{c} M_{2,q} \\ \text{---} \\ \text{---} \\ \text{---} \end{array}}{E \cdot J_2} dx$$

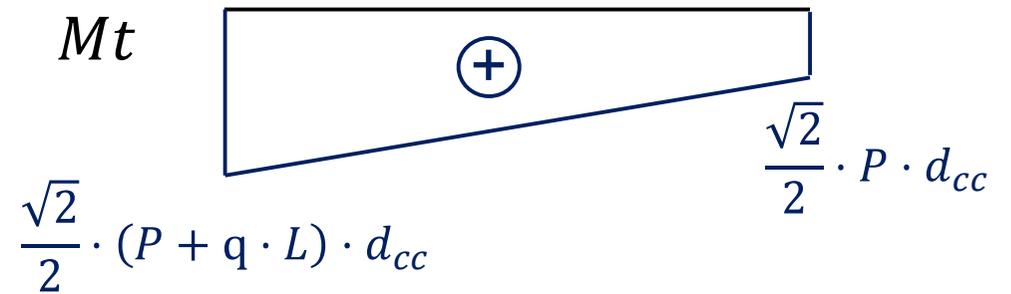
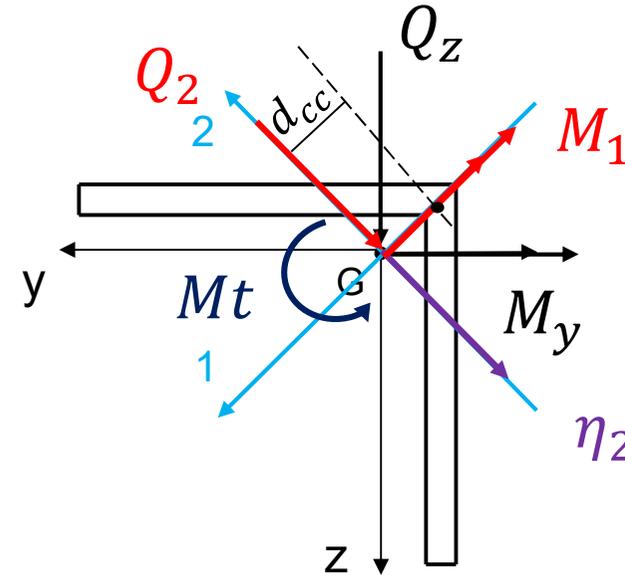
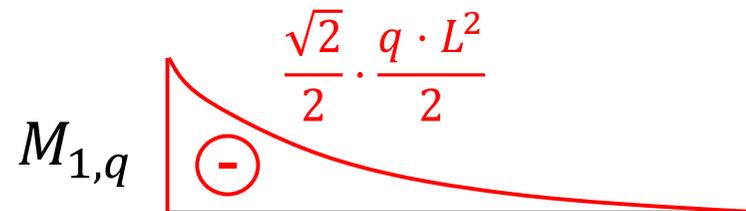
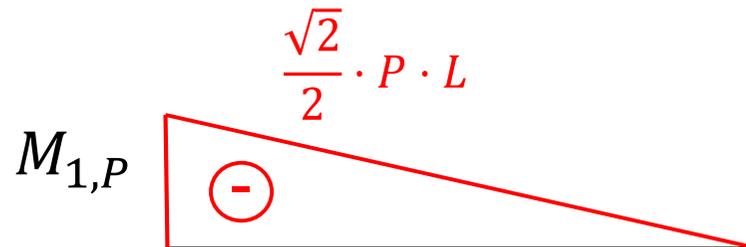
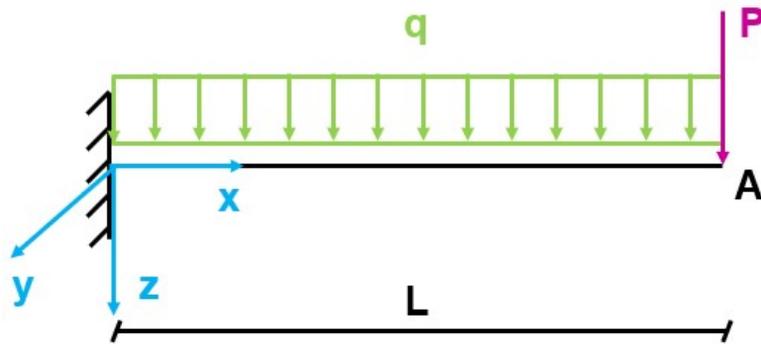
$$(+1) \cdot \eta_{A,1} = \frac{1}{3} \cdot (-(+1) \cdot L) \cdot \frac{\sqrt{2}}{2} \cdot \frac{(-P \cdot L)}{E \cdot J_2} \cdot L + \frac{1}{4} \cdot (-(+1) \cdot L) \cdot \frac{\sqrt{2}}{2} \cdot \frac{(-q \cdot L^2)}{2 \cdot E \cdot J_2} \cdot L$$

$$\eta_{A,1} = \frac{\sqrt{2}}{2 \cdot E \cdot J_2} \cdot \left(\frac{P \cdot L^3}{3} + \frac{q \cdot L^4}{8} \right)$$



Desplazamiento η_2

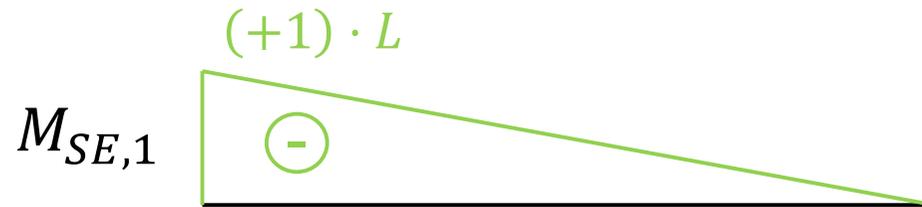
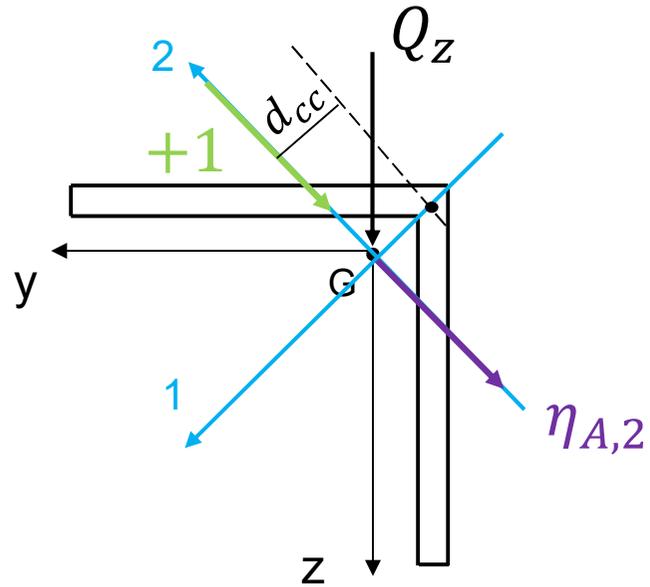
- Diagramas de deformación virtual





- Sistema equilibrado → Aplicamos fuerza unitaria en **A** con dirección 2

- Sección A





- Aplicamos el **Teorema de los trabajos virtuales**

$$(+1) \cdot \eta_{A,2} = \underbrace{\int \frac{M_1^{SE} \cdot M_1^{DV}}{E \cdot J_1} dx}_{\textcircled{1}} + \underbrace{\int \frac{M_t^{SE} \cdot M_t^{DV}}{G \cdot J} dx}_{\textcircled{2}}$$

$$\textcircled{1} = \int_0^L \frac{(+1) \cdot L \cdot M_{1,p}}{E \cdot J_1} dx + \int_0^L \frac{(+1) \cdot L \cdot M_{1,q}}{E \cdot J_1} dx$$

$$\textcircled{1} = \frac{\sqrt{2}}{2 \cdot E \cdot J_1} \cdot \left(\frac{P \cdot L^3}{3} + \frac{q \cdot L^4}{8} \right)$$



$$\textcircled{2} = \int_0^L \frac{(+1) \cdot d_{cc} \cdot \frac{\sqrt{2}}{2} \cdot (P + q \cdot L) \cdot d_{cc} \cdot \frac{\sqrt{2}}{2} \cdot P \cdot d_{cc}}{G \cdot J} dx = \frac{\sqrt{2}}{4 \cdot G \cdot J} \cdot d_{cc}^2 \cdot L \cdot (2 \cdot P + q \cdot L)$$

↑
Aplicando la fórmula
y reordenando

$$\boxed{M \cdot \bar{M}_A \bar{M}_B = \frac{1}{2} \cdot M \cdot (\bar{M}_A + \bar{M}_B) \cdot L}$$

- Entonces,

$$\eta_{A,2} = \textcircled{1} + \textcircled{2}$$

$$\eta_{A,2} = \frac{\sqrt{2}}{2 \cdot E \cdot J_1} \cdot \left(\frac{P \cdot L^3}{3} + \frac{q \cdot L^4}{8} \right) + \frac{\sqrt{2}}{4 \cdot G \cdot J} \cdot d_{cc}^2 \cdot L \cdot (2 \cdot P + q \cdot L)$$



- Cual es el **desplazamiento máximo** del punto A?

$$\eta_{A,1} = \frac{\sqrt{2}}{2 \cdot E \cdot J_2} \cdot \left(\frac{P \cdot L^3}{3} + \frac{q \cdot L^4}{8} \right)$$

$$\eta_{A,2} = \frac{\sqrt{2}}{2 \cdot E \cdot J_1} \cdot \left(\frac{P \cdot L^3}{3} + \frac{q \cdot L^4}{8} \right) + \frac{\sqrt{2}}{4 \cdot G \cdot J} \cdot d_{cc}^2 \cdot L \cdot (2 \cdot P + q \cdot L)$$

- El **desplazamiento máximo** del punto A, considerando esta sección, va a ser la composición de los **calculados en los ejes 1 y 2**

$$\eta_A = \sqrt{(\eta_{A,1})^2 + (\eta_{A,2})^2}$$

¡Observación!
La dirección del desplazamiento
no es z