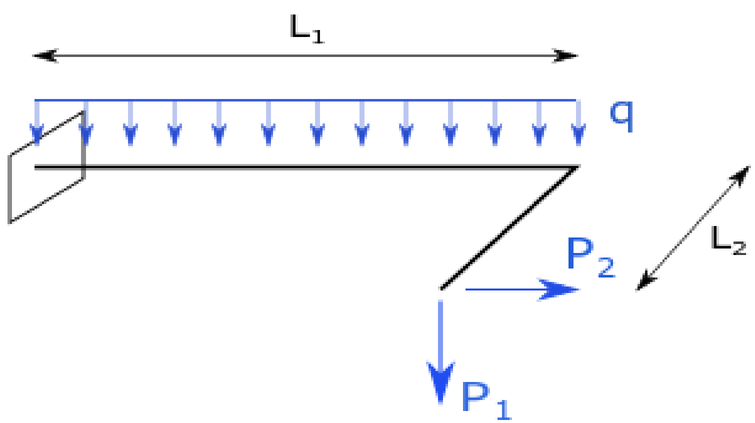
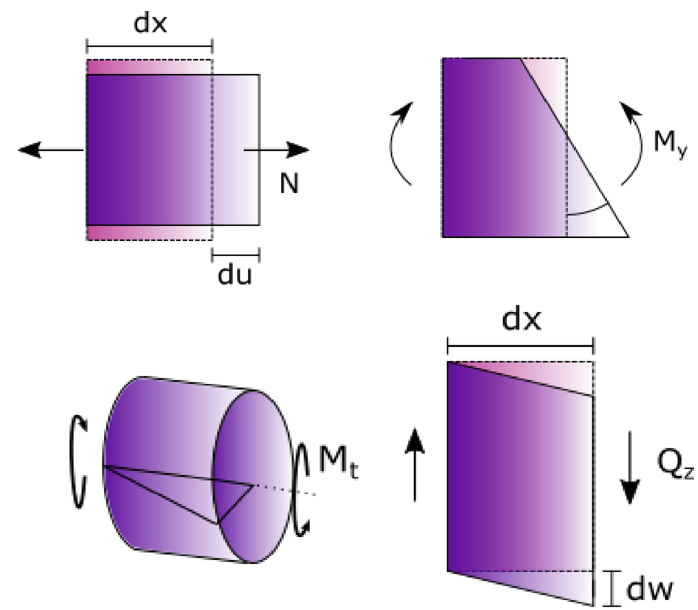
# Cálculo de desplazamientos

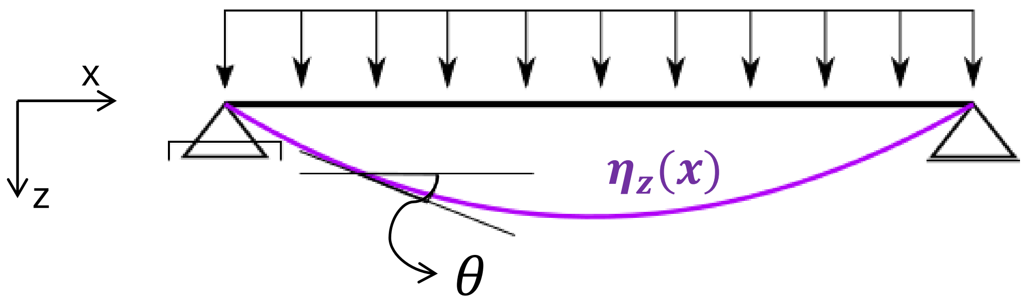


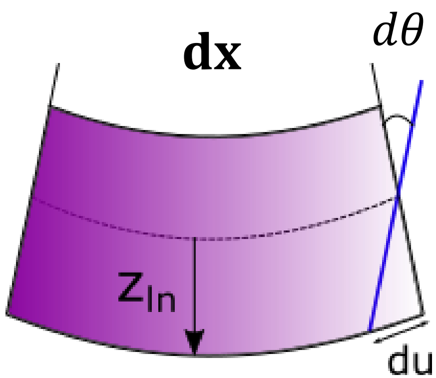

Tania Poletilo - Constanza Ruffinelli - Manuela Medina

# Repaso teórico

## Desplazamiento por flexión y corte




$$\theta \approx tg(\theta) = -\frac{d\eta_z(x)}{dx}$$




Sabemos que $du = \varepsilon_x \cdot dx$

A partir del esquema: $du = d\theta \cdot z_{ln}$

$$d\theta \cdot z_{ln} = \varepsilon_x \cdot dx \longrightarrow \frac{d\theta}{dx} = \chi = \frac{\varepsilon_x}{z_{ln}}$$

Recordando la Ley de Hooke $(\sigma = E \cdot \varepsilon)$

y el cálculo de sigma para EPI $\left(\sigma = \frac{M_y}{J_y} \cdot z\right)$

$$\frac{d^2\eta_z(x)}{dx^2} = -\frac{d\theta}{dx} = -\frac{\sigma_x}{E \cdot z_{ln}} = -\frac{M_y \cdot z_{ln}}{J_y \cdot E \cdot z_{ln}} = -\frac{M_y}{J_y \cdot E}$$

# Ecuación diferencial de la elástica de deformación

$$\eta_z$$

$$-\theta = \frac{d\,\eta_z(x)}{dx}$$

$$-\chi = \frac{d^2\eta_z(x)}{dx^2} = -\frac{M_y(x)}{J_y \cdot E}$$

$$\frac{d^3\eta_z(x)}{dx^3} = -\frac{1}{J_y \cdot E} \cdot \frac{dM_y(x)}{dx} = -\frac{Q_z(x)}{J_y \cdot E}$$

$$\frac{d^4\eta_z(x)}{dx^4} = -\frac{1}{J_y \cdot E} \cdot \frac{dQ_z(x)}{dx} = \frac{q_z(x)}{J_y \cdot E}$$

# Entonces

Si $q = cte \Rightarrow Q: lineal \Rightarrow M: cuadrática \Rightarrow \theta$: cúbica

$\eta_z(x): polinomio\ de\ orden\ 4 \Rightarrow \eta_z(x) = A \cdot x^4 + B \cdot x^3 + C \cdot x^2 + D \cdot x + F$

$$\frac{d\eta_z(x)}{dx} = -\theta = 4 \cdot A \cdot x^3 + 3 \cdot B \cdot x^2 + 2 \cdot C \cdot x + D$$

$$\frac{d^2\eta_z(x)}{dx^2} = -\chi = -\frac{M_y(x)}{J_y \cdot E} = 12 \cdot A \cdot x^2 + 6 \cdot B \cdot x + 2 \cdot C$$

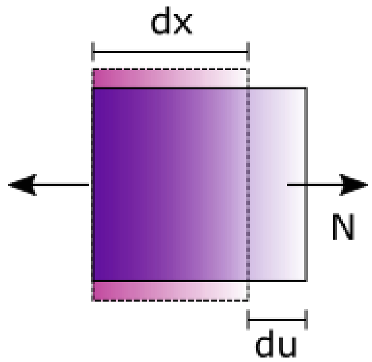
$$\frac{d^3\eta_z(x)}{dx^3} = -\frac{Q_z(x)}{J_y \cdot E} = 24 \cdot A \cdot x + 6 \cdot B$$

$$\frac{d^4\eta_z(x)}{dx^4} = \frac{q_z}{J_y \cdot E} = 24 \cdot \mathrm{A} \Rightarrow A = \frac{q_z}{24 \cdot J_y \cdot E}$$

Como tengo que determinar 4 constantes (B, C, D, F), necesito 4 condiciones de borde

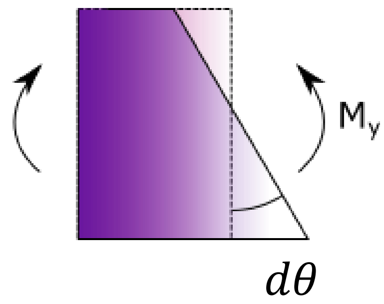
# Relaciones entre solicitaciones y deformada

## Axil



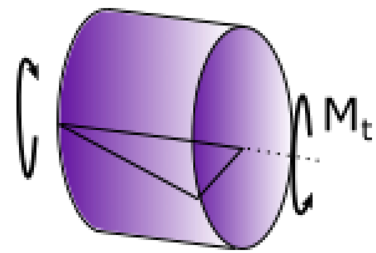

$$du = \varepsilon_x \cdot dx = \frac{N}{E \cdot A}\,dx$$

## Flexión



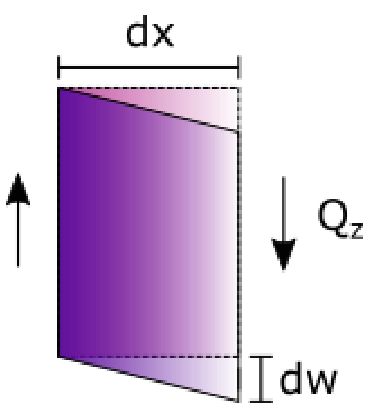

$$-d\theta_y = \chi_y\,dx = \frac{M_y}{J_y \cdot E}\,dx$$

## Torsión




$$d\theta_x = \chi_x\,dx = \frac{M_t}{J_p \cdot G}\,dx$$

## Corte




$$dw = K_y \cdot \frac{Q_z}{G \cdot A}\,dx$$

$$K_y = \int_A \frac{S^2}{i_y^4 \cdot A \cdot b^2}\,dA$$

Con $K_y$: factor de forma

Para barras donde la $L >> a$ (L: longitud de la barra, a: mayor dimensión de la sección), se desprecia la deformación por corte

# Cálculo de desplazamientos por TTV

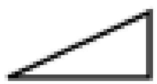
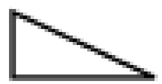
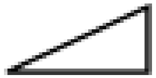
$$W_E = W_I$$

$$(+1)\cdot\eta = \int_L N_{se}\, du_{dv} + \int_L M_{y_{se}}\, d\theta_{y_{dv}} + \int_L M_{t_{se}}\, d\theta_{x_{dv}} + \int_L Q_{se}\, dw_{dv}$$

$$(+1)\cdot\eta = \int_L N^{se}\cdot\frac{N^{dv}}{E\cdot A}\,dx + \int_L M_y^{se}\cdot\frac{M_y^{dv}}{J_y\cdot E}\,dx + \int_L M_t^{se}\cdot\frac{M_t^{dv}}{J_p\cdot G}\,dx$$

$$\int i(x)\cdot j(x)\cdot dx = \alpha\cdot\ i\cdot j\cdot L$$

$i$ y $j$ valores máximos de la función

| $\alpha$ |  |  |  |  |
|---|---|---|---|---|
|  | 1 | $^1/_2$ | $^1/_2$ | $^2/_3$ |
|  | $^1/_2$ | $^1/_3$ | $^1/_6$ | $^1/_3$ |

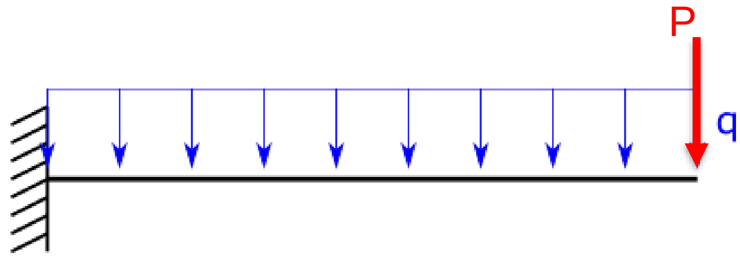
Allí la pendiente es cero

**Aclaración**

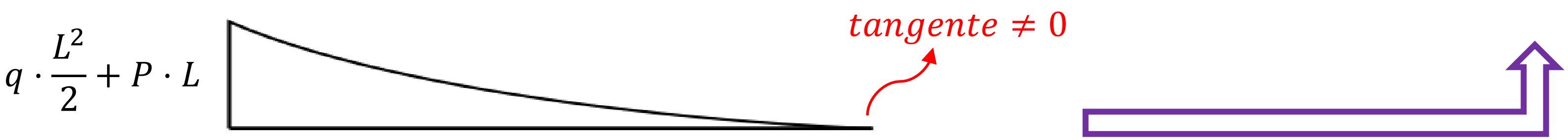
Ésta es una tabla simple, hay otras con más combinaciones pero siempre se pueden descomponer los diagramas en una suma de éstas

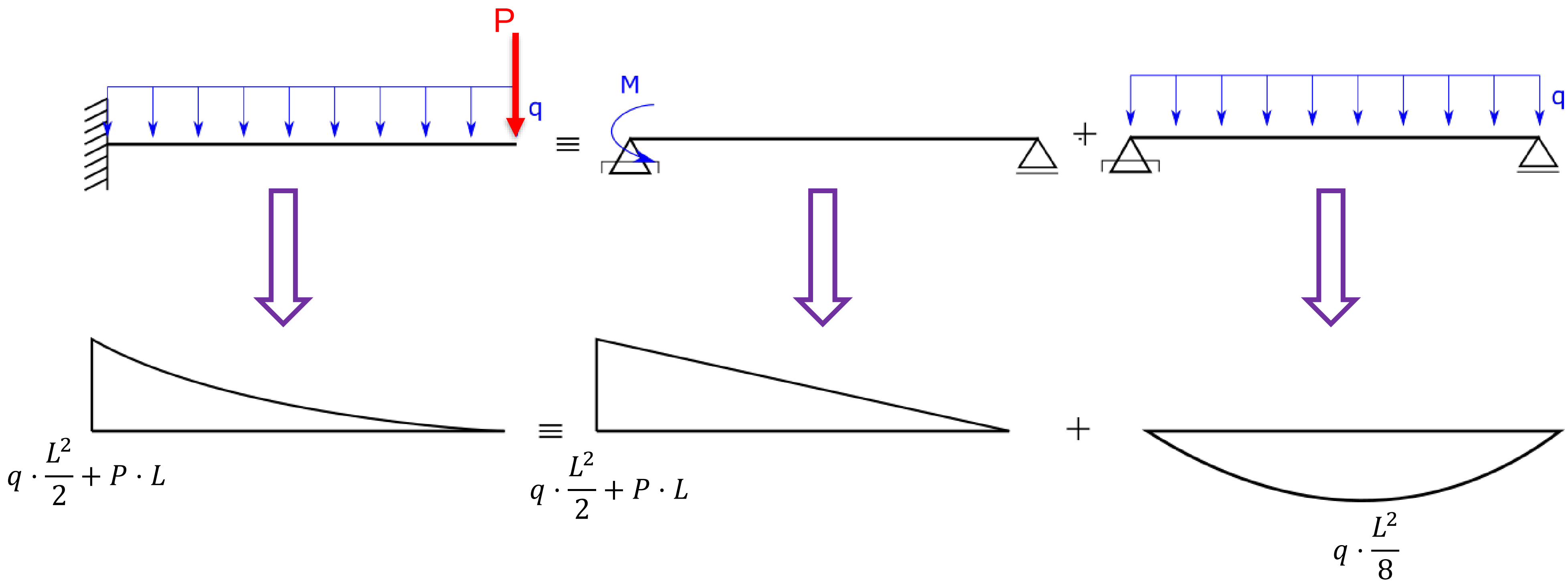
https://campus.fi.uba.ar/mod/folder/view.php?id=43794

# Descomposición de diagramas

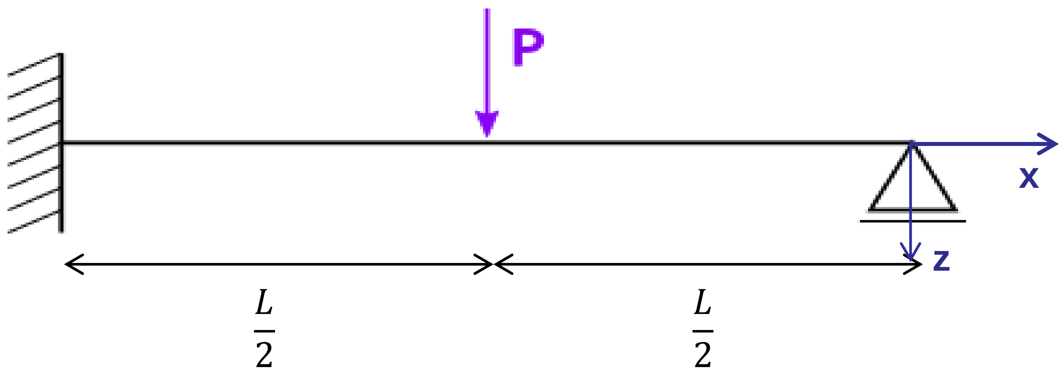



Cómo ésta función no la encuentro en tabla debo dividirla en suma de funciones que si estén



# Ejercicio 1: Resolución de la elástica por ecuación diferencial y condiciones de borde




¿Cuántas funciones necesito?

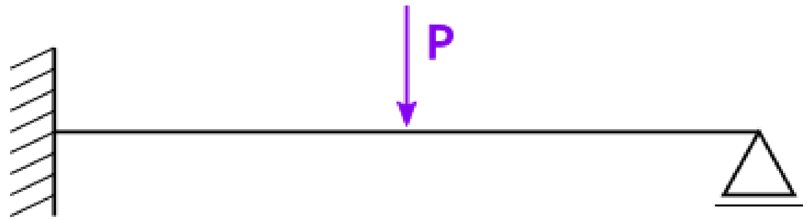
¿Cuántas condiciones de borde necesito?

Cada función $w_i(x)$ tiene 4 incógnitas

$$w(x) = \begin{cases} w_1(x)\ si\ 0 < x < \frac{L}{2} \\ w_2(x)\ si\ \frac{L}{2} < x < L \end{cases}$$

⇒ 8 condiciones de vínculo

# Condiciones de borde




**1)** $w_1(x = 0) = 0$

**2)** $\theta_1(x = 0) = 0$

**3)** $w_2(x = L) = 0$

**4)** $M(x = L) = 0 \rightarrow \chi_2(x = L) = 0$

**5)** $w_1\left(x = \frac{L}{2}\right) = w_2\left(x = \frac{L}{2}\right)$

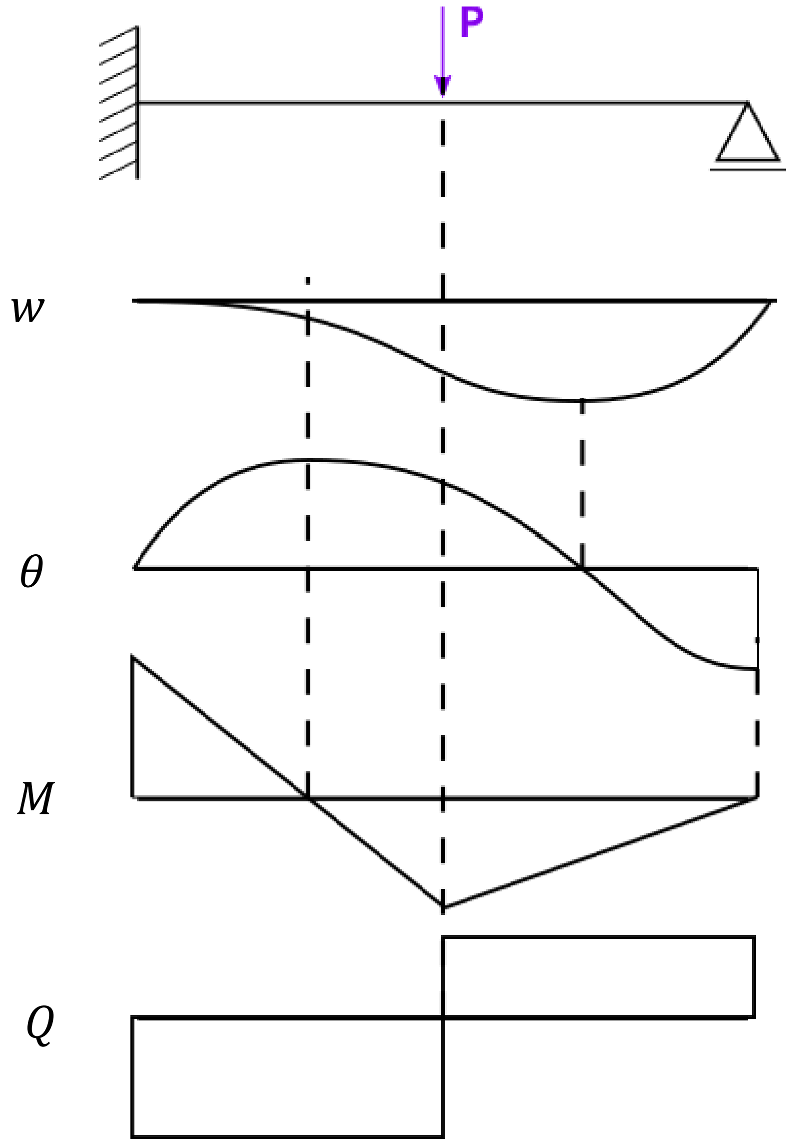
**6)** $\theta_1\left(x = \frac{L}{2}\right) = \theta_2\left(x = \frac{L}{2}\right)$

**7)** $M_1\left(x = \frac{L}{2}\right) = M_2\left(x = \frac{L}{2}\right)$

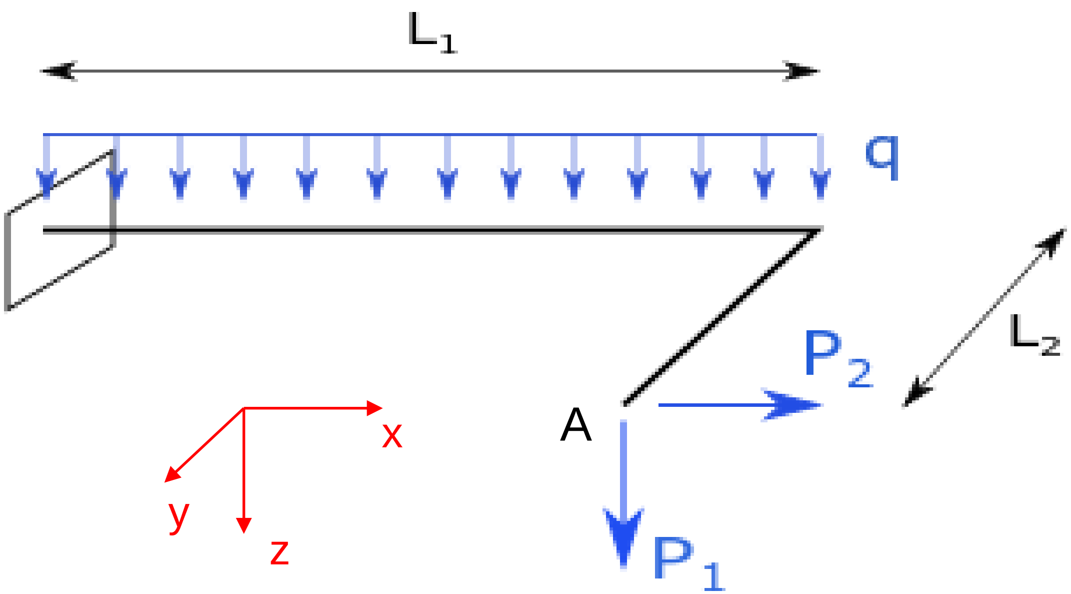
**8)** $Q_1\left(x = \frac{L}{2}\right) = Q_2\left(x = \frac{L}{2}\right) + P$

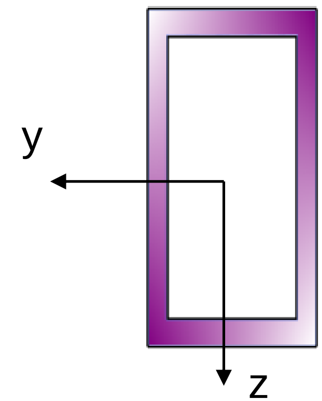
Una vez determinadas las 8 condiciones de borde tenemos un sistema de 8 ecuaciones y 8 incógnitas.

# Diagramas



## Ejercicio 2: Calcular el desplazamiento vertical en A y $\theta_{Ax}$.




$Perfil\text{: } 100\ x\ 250\ x\ 10\ mm$

$J_y = 4515\ \text{cm}^4$

$J_z = 1040\ \text{cm}^4$

$J_t = 2777\ \text{cm}^4$

Cargas:

$P_1 = 5\ kN$ $\qquad L_1 = 4\ m$

$P_2 = 10\ kN$ $\qquad L_2 = 1\ m$

$q = 5\ \dfrac{kN}{m}$

$E = 21000\ \dfrac{kN}{cm^2}$ $\qquad G = 8000\ \dfrac{kN}{cm^2}$

# Resolución:

Calculamos cada desplazamiento con TTV

$$W_E = W_i$$

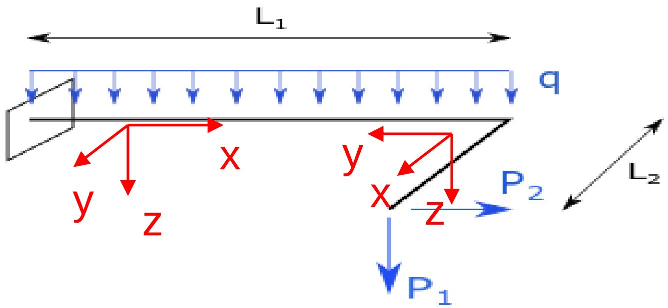
$$(+1) \cdot \eta = \int_L N^{se} \cdot \frac{N^{dv}}{E \cdot A}\, dx + \int_L M_{en}^{se} \cdot \frac{M_{en}^{dv}}{J_{en} \cdot E}\, dx + \int_L M_t^{se} \cdot \frac{M_t^{dv}}{J_t \cdot G}\, dx$$

Por lo que necesitamos:

- Diagramas del DV
- Diagramas de cada SE

# Diagramas de características

*DV*

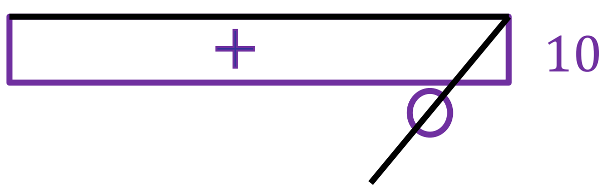



$P_1 = 5\ kN$ $L_1 = 4\ m$

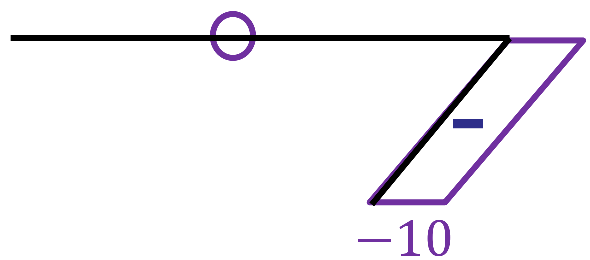
$P_2 = 10\ kN$ $L_2 = 1\ m$

$q = 5\ \frac{kN}{m}$

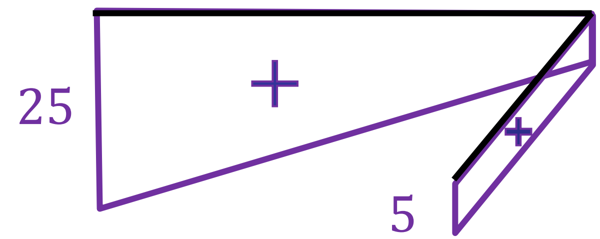
$N[kN]$



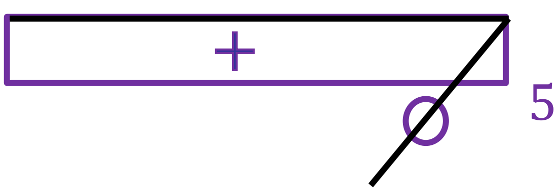

$Q_y[kN]$



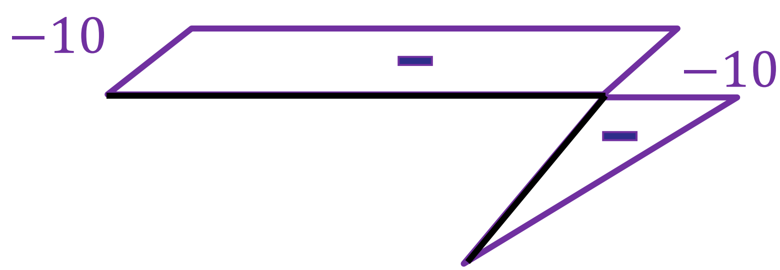

$Q_z[kN]$



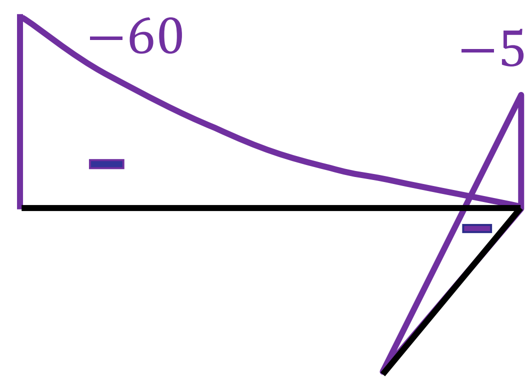

$M_t[kN\ m]$



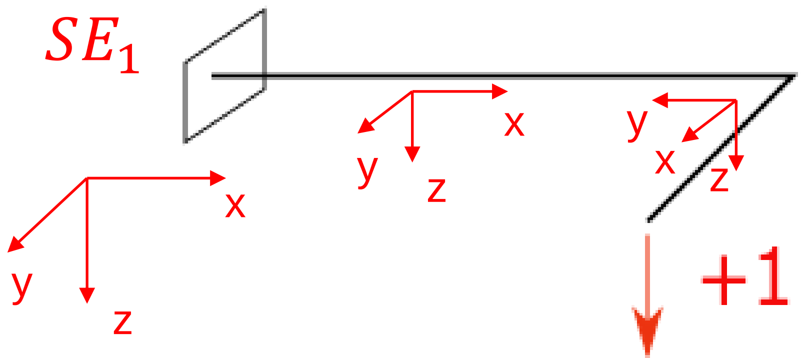

$M_z[kN\ m]$



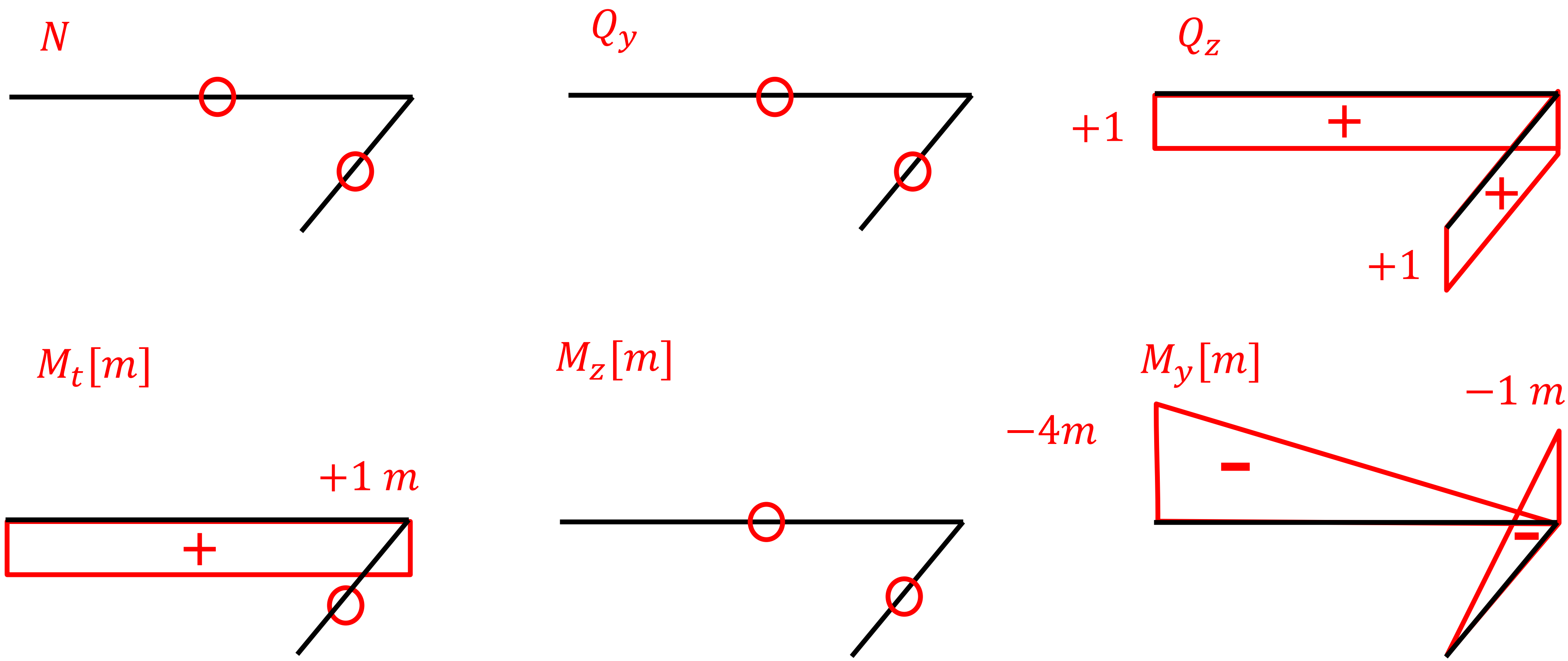

$M_y[kN\ m]$



# Desplazamiento vertical: $\eta_A$

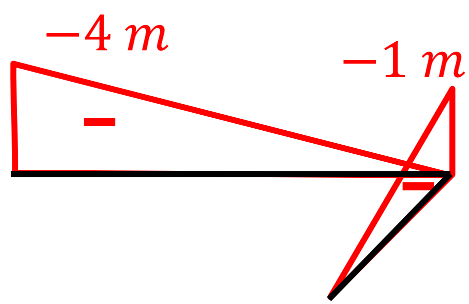



Quiero un desplazamiento vertical, por lo que en el SE debo poner una fuerza unitaria vertical en A

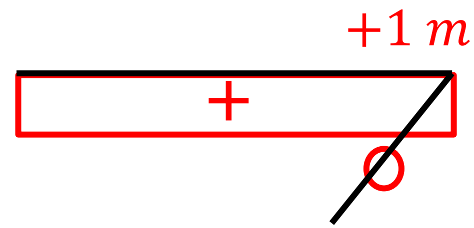


$SE_1$

$M_y[m]$

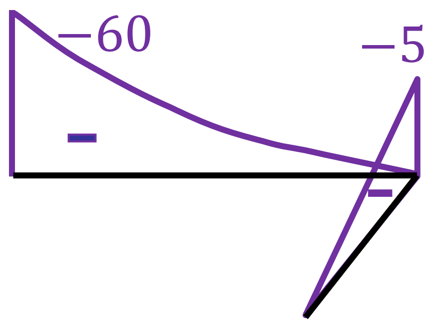



$M_t[m]$

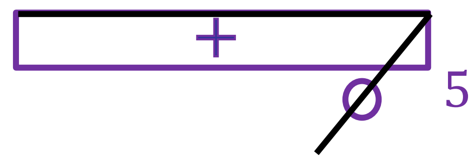



$DV$

$M_y[kN\ m]$

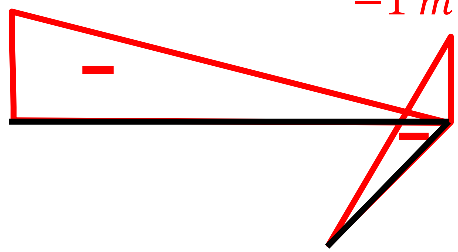


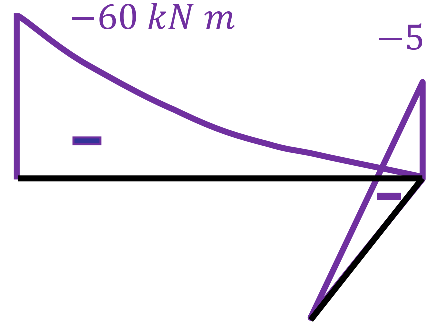

$M_t[kN\ m]$




$$\eta_A = \int_L M_{se} \cdot \frac{M_{dv}}{E \cdot J}\, dx + \int_L Mt_{se} \cdot \frac{Mt_{dv}}{G \cdot J_t}\, dx = \eta_A^{M_y} + \eta_A^{M_t}$$

$$\eta_A^{M_y}$$

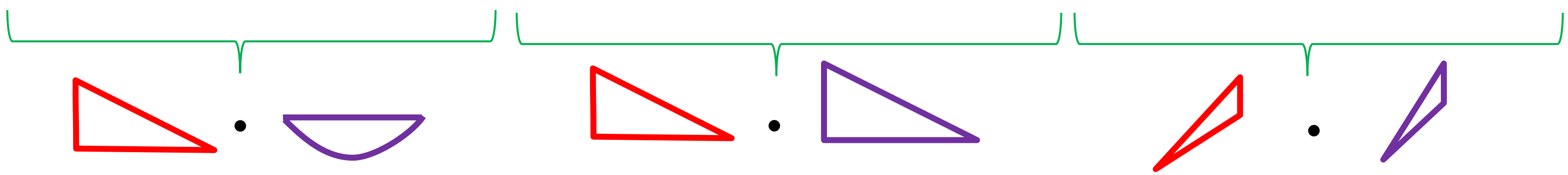



Cuidado!
La tangente no es nula, por lo tanto el coeficiente no esta en tabla.


$$\frac{q \cdot L^2}{8} = 10\ kN\ m$$

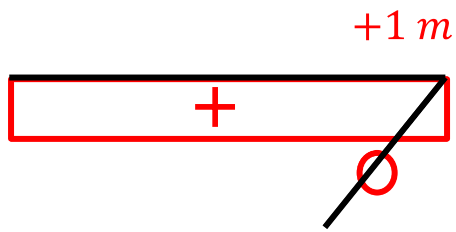
$$\int_L M_{se} \cdot \frac{M_{dv}}{E \cdot J} dx = \frac{1}{3} \cdot (-4m) \cdot \frac{10kNm}{E \cdot J_y} \cdot 4m + \frac{1}{3} \cdot (-4m) \cdot \frac{(-60kNm)}{E \cdot J_y} \cdot 4m + \frac{1}{3} \cdot (-1m) \cdot \frac{(-5kNm)}{E \cdot J_y} \cdot 1m$$

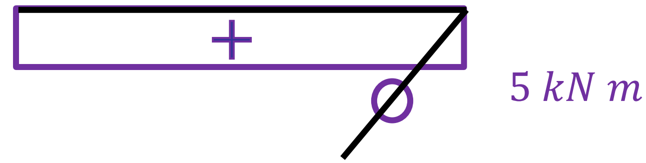


$$\eta_A^{M_y} = -0{,}562\ cm + 3{,}375\ cm + 0{,}018\ cm$$

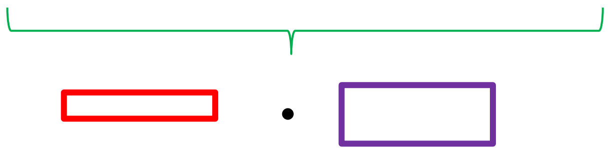
$$\eta_A^{M_y} = 2{,}831\ cm$$

$\eta_A^{M_t}$




$$\int_L Mt_{se} \cdot \frac{Mt_{dv}}{G \cdot J_t} dx = 1 \cdot (+1\ m) \cdot \frac{(+5\ kN\ m)}{G \cdot J_t} \cdot 4\ m$$



$$\eta_A^{M_t} = +0{,}900\ cm$$

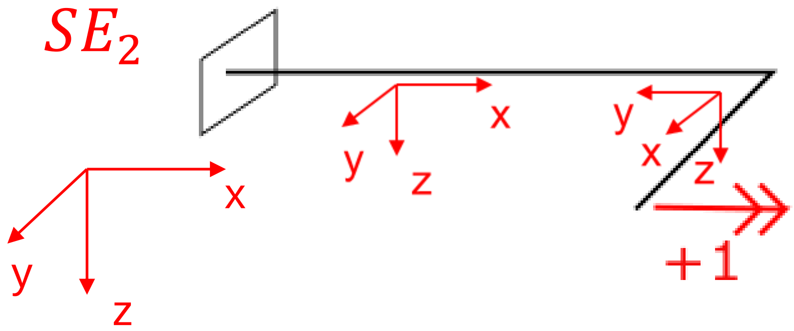
Desplazamiento total:

$$\eta_A = \eta_A^{M_y} + \eta_A^{M_t}$$

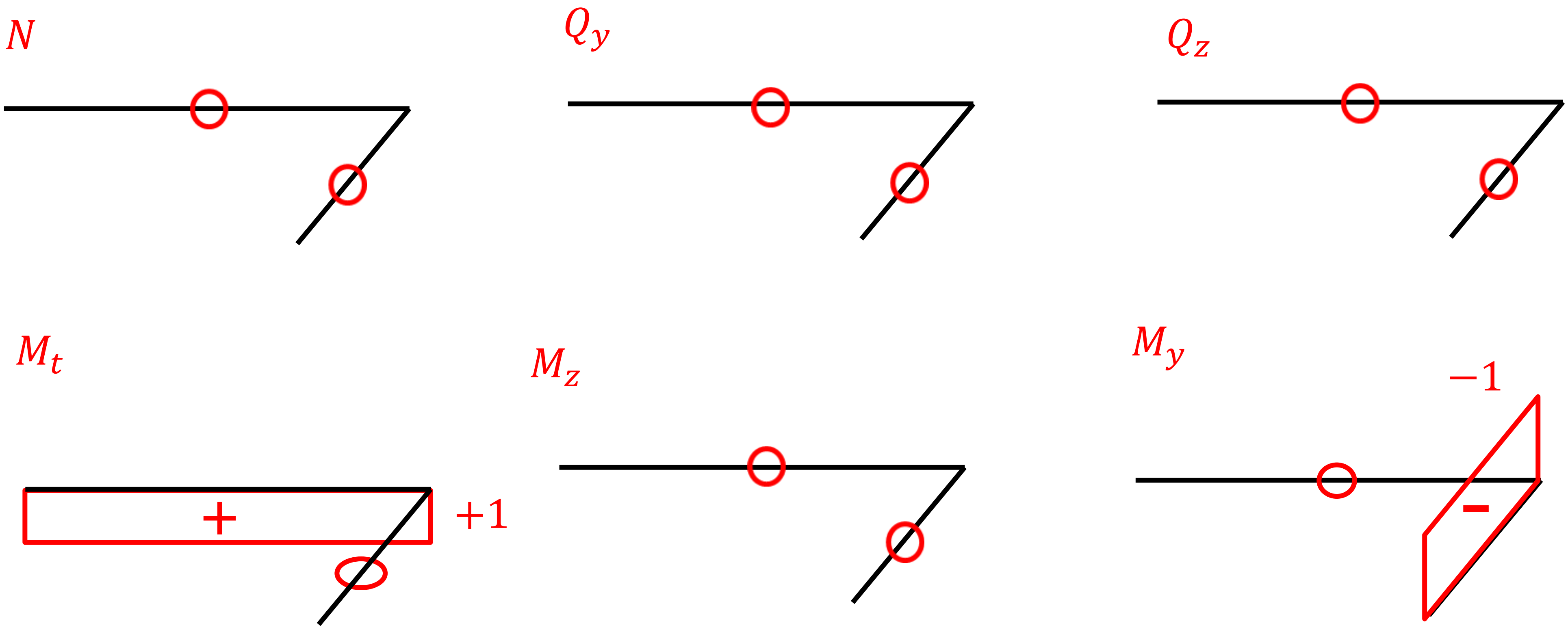
$$\eta_A = 2{,}831\ cm + 0{,}900\ cm$$

$$\boxed{\eta_A = 3{,}731\ cm}$$

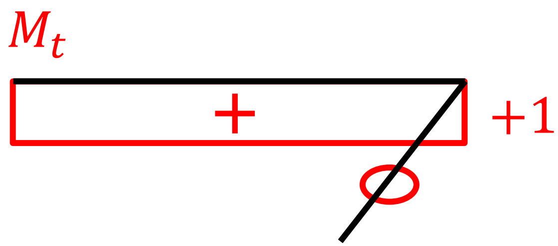
# Giro en A: $\theta_{xA}$

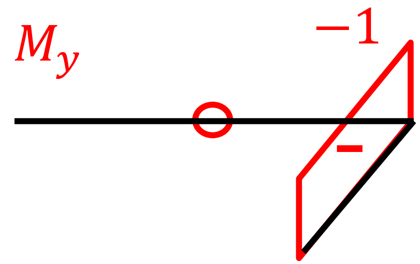



Quiero un giro, por lo que en el SE debo poner un momento unitario en A

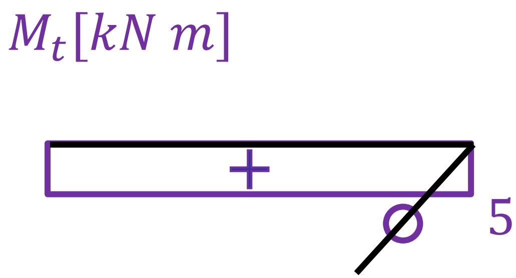


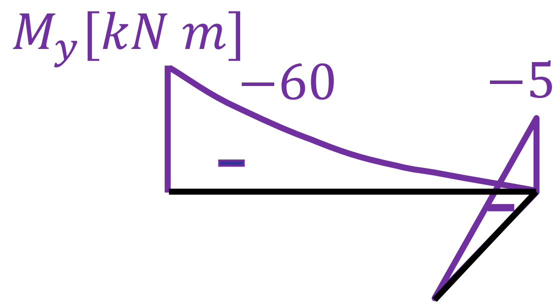
$SE_1$




$DV$




$$\theta_{xA} = \int_L Mt_{se} \cdot \frac{Mt_{dv}}{G \cdot J_t} dx + \int_L M_{se} \cdot \frac{M_{dv}}{E \cdot J} dx$$

$$\theta_{xA} = \underbrace{1 \cdot (+1) \cdot \frac{(+5\ kN\ m)}{G \cdot J_t} \cdot 4\ m}_{} + \underbrace{\frac{1}{2} \cdot (-1) \cdot \frac{(-5\ kN\ m)}{E \cdot J_y} \cdot 1\ m}_{}$$

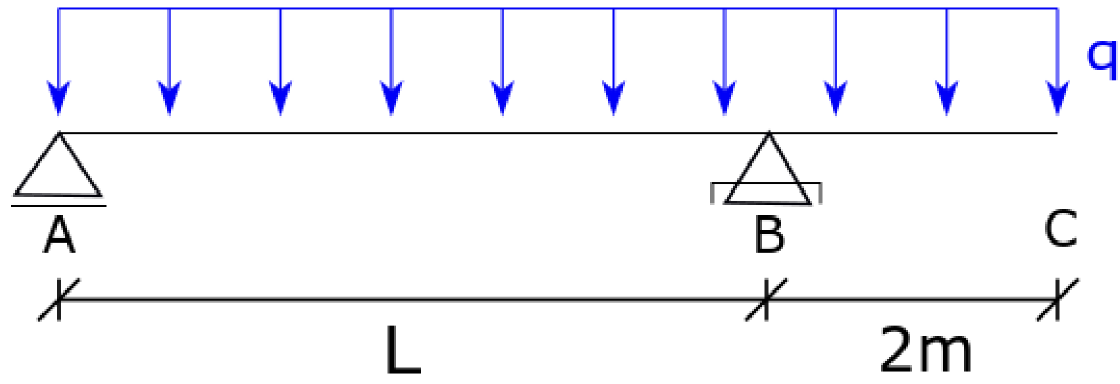
$$\theta_{xA} = 9{,}00 \cdot 10^{-3}\ rad + 2{,}63 \cdot 10^{-4}\ rad$$

$$\boxed{\theta_{xA} = 9{,}263 \cdot 10^{-3}\ \text{rad}}$$

# Ejercicio 3:

**a)** Calcular la longitud “L” tal que el giro en la sección “B” sea nulo

**b)** Calcular el desplazamiento en “C”

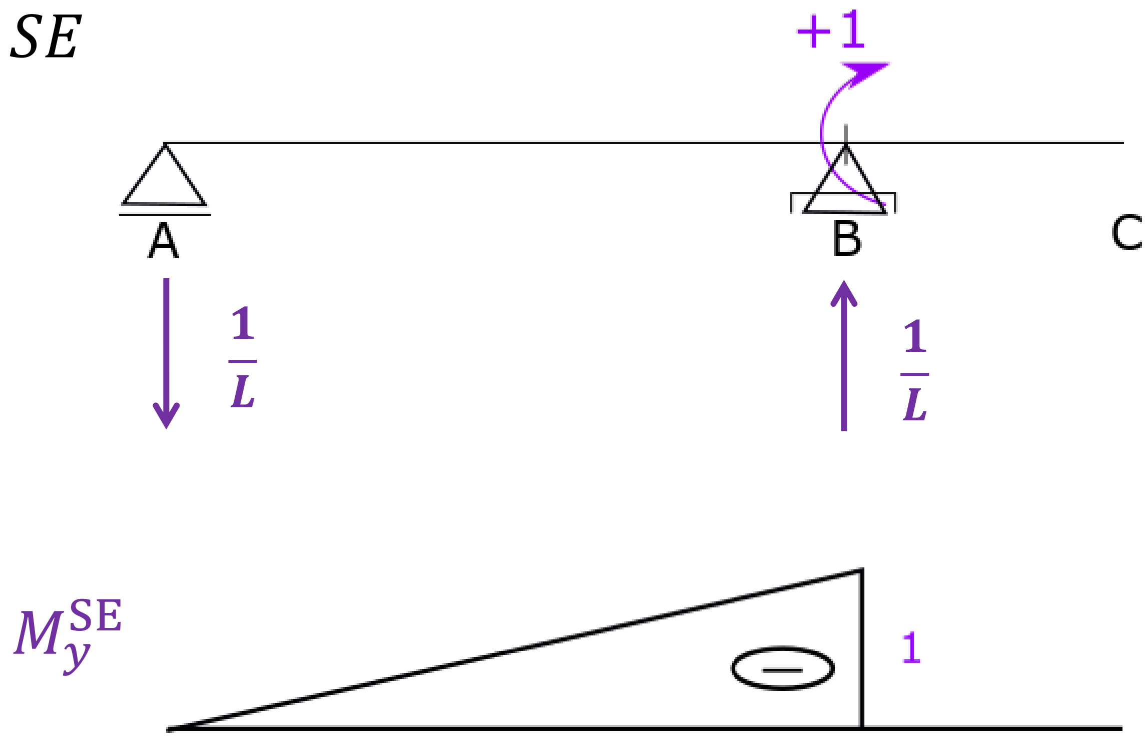



**Datos:** $E \cdot J = cte$

$$q = 10\ \frac{kN}{m}$$

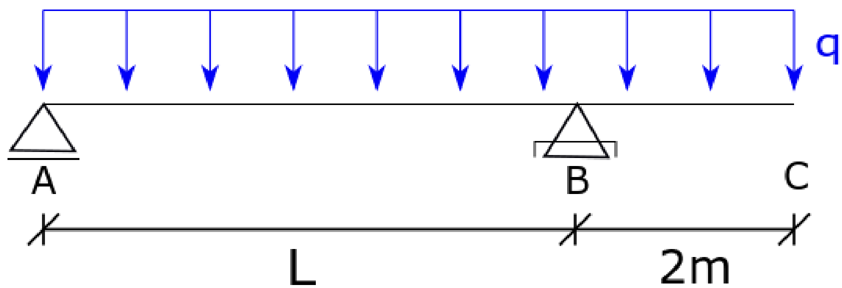
# Calculo el giro utilizando TTV

Quiero calcular el giro, por lo tanto en el sistema equilibrado tengo que poner un momento unitario en “B”



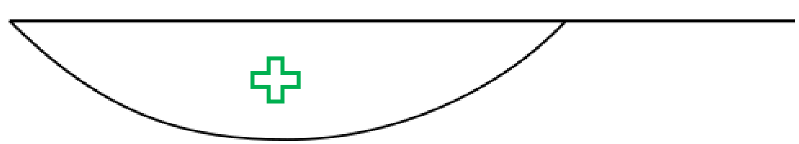
# $DV$

Opción 1: Realizo los diagramas de características directamente de la estructura completa




Opción 2: Utilizo superposición de efectos

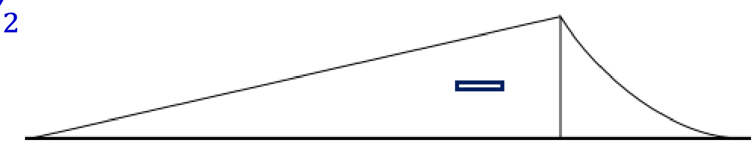

$DV_1$



$q \cdot \frac{L^2}{8}$

$DV_2$

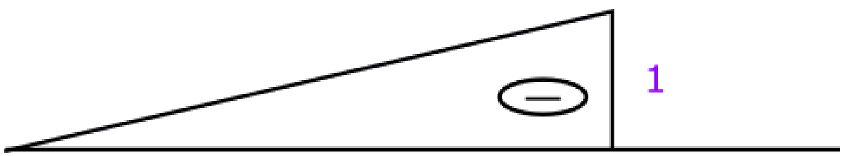
$20\ kN\ m$

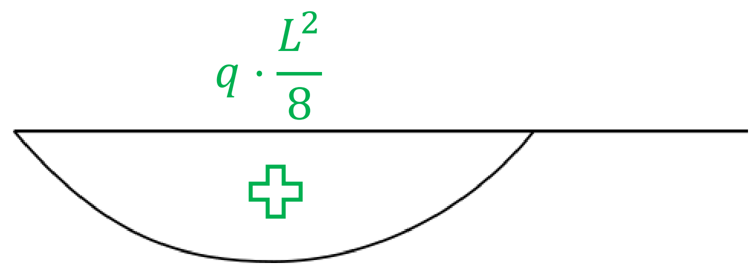


$$\theta_B = \int_L M^{SE} \cdot \frac{M^{DV}}{E \cdot J}\, dx = \int M^{SE} \cdot \frac{M^{DV_1}}{E \cdot J}\, dx + \int M^{SE} \cdot \frac{M^{DV_2}}{E \cdot J}\, dx$$

$$\int M^{SE} \cdot \frac{M^{DV_1}}{E \cdot J}\, dx \Longrightarrow$$

| $\alpha$ | | | | |
|---|---|---|---|---|
| | 1 | $^1/_2$ | $^1/_2$ | $^2/_3$ |
| | $^1/_2$ | $^1/_3$ | $^1/_6$ | $^1/_3$ |

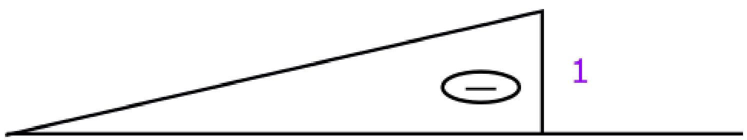


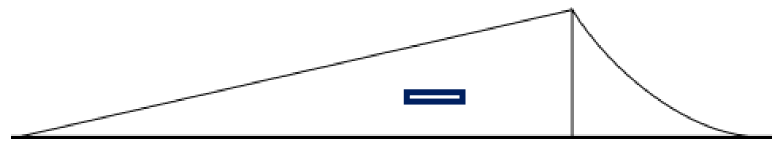




$$\int M^{SE} \cdot \frac{M^{DV_1}}{E \cdot J}\, dx = \frac{1}{3} \cdot \frac{(-1) \cdot \left(10\, \frac{kN}{m} \cdot \frac{L^2}{8}\right) \cdot L}{E \cdot J} = -\frac{5}{12}\, \frac{kN}{m} \cdot \frac{L^3}{E \cdot J}$$

$$\int M^{SE} \cdot \frac{M^{DV_2}}{E \cdot J}\, dx \Longrightarrow$$

| $\alpha$ | | | | |
|---|---|---|---|---|
| | 1 | $^1/_2$ | $^1/_2$ | $^2/_3$ |
| | $^1/_2$ | $^1/_3$ | $^1/_6$ | $^1/_3$ |




$$\int M^{SE} \cdot \frac{M^{DV_2}}{E \cdot J}\, dx = \frac{1}{3} \cdot \frac{(-1) \cdot (-20\ kN\ m) \cdot L}{E \cdot J} = \frac{20}{3} \cdot \frac{L}{E \cdot J}\ kN\ m$$

$$\theta_B = \int M^{SE} \cdot \frac{M^{DV_1}}{E \cdot J}\, dx + \int M^{SE} \cdot \frac{M^{DV_2}}{E \cdot J}\, dx$$

Reemplazando y recordando que $\theta_B = 0$

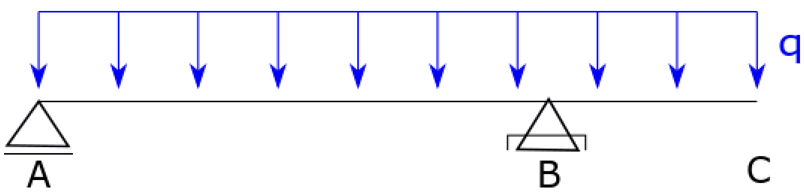
$$0 = -\frac{5}{12} \frac{kN}{m} \cdot \frac{L^3}{E \cdot J} + \frac{20}{3} \cdot \frac{L}{E \cdot J}\, kN\, m$$

$$\frac{5}{12} \frac{kN}{m} \cdot \frac{L^3}{E \cdot J} = \frac{20}{3} \cdot \frac{L}{E \cdot J}\, kN\, m \rightarrow \frac{5}{12} \cdot L^2 = \frac{20}{3}\, m^2$$

$$\boxed{L = 4\, m}$$

## **b)** Desplazamiento en “C”

*DV*

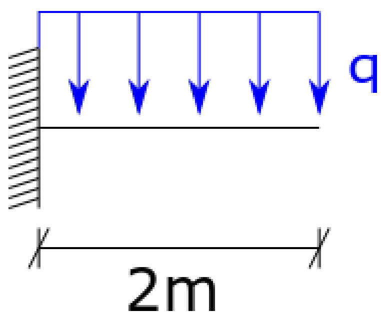



*SE*


Pero sabemos que en el punto B no hay giros ni desplazamientos

El punto B funciona como un empotramiento

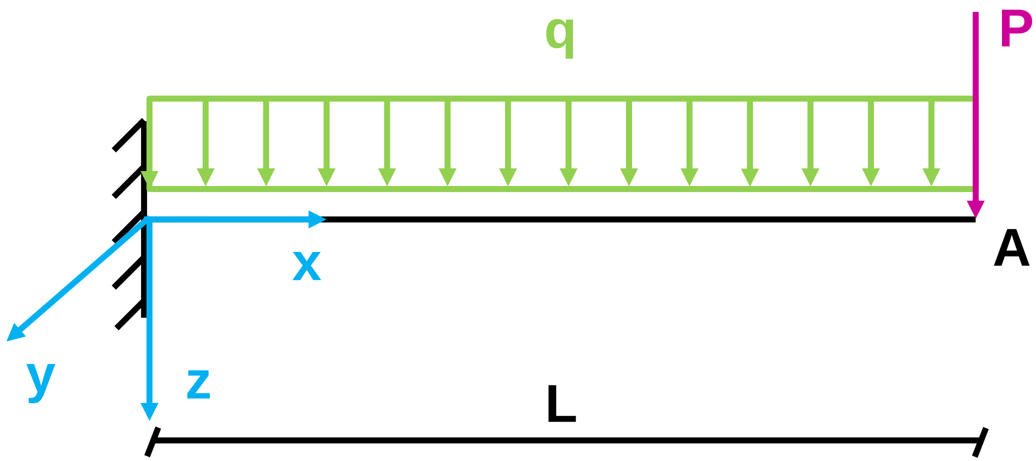
*DV*



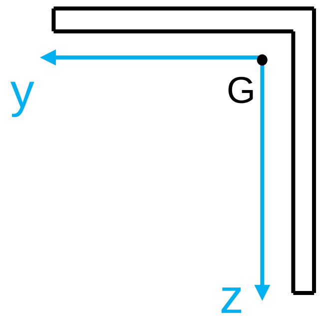

*SE*

# Ejercicio 4:

**a)** Calcular el desplazamiento máximo del punto A




Datos



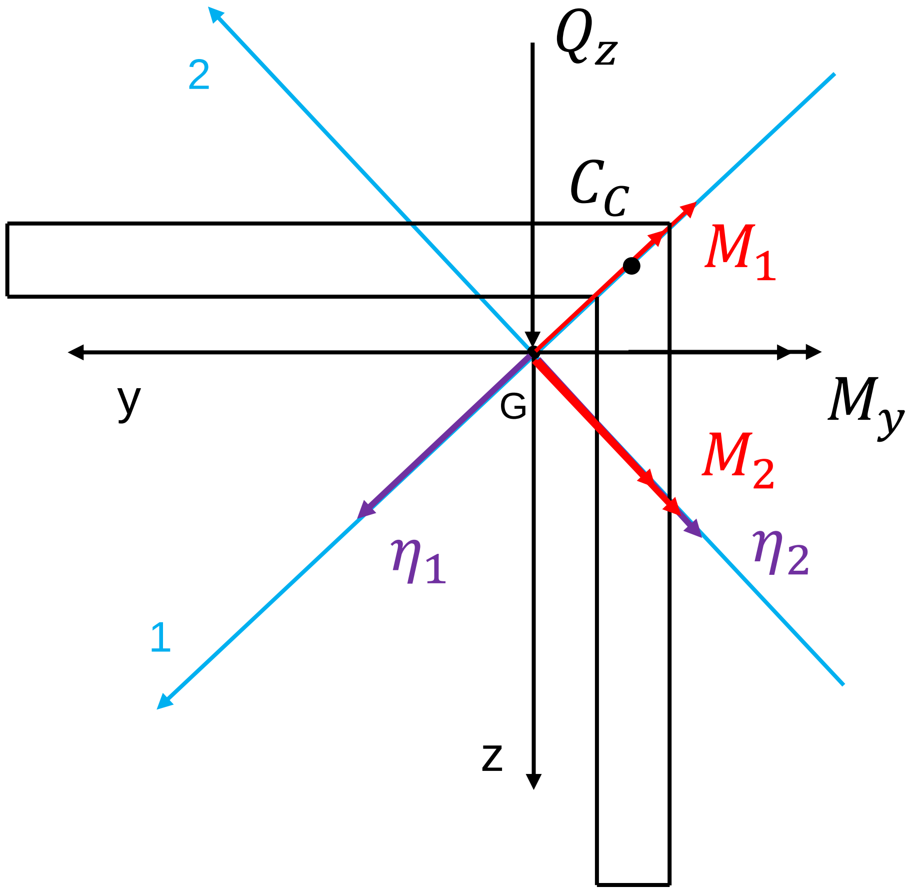

$L$

$P, q$

$E$

$Perfil\ angulo\ de\ alas\ iguales$

- Analizamos la sección



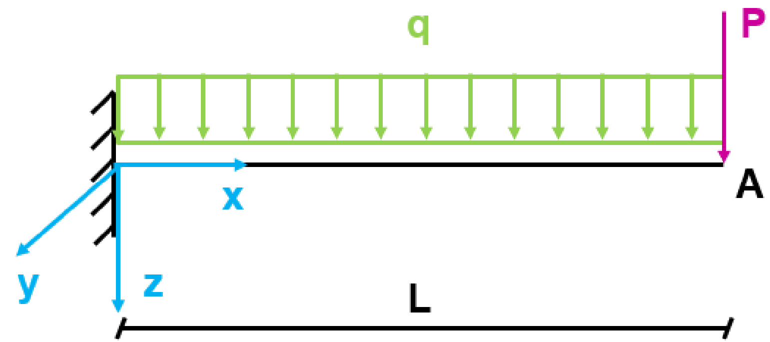

**¡Observación!**
Tenemos que trabajar con los ejes principales de inercia

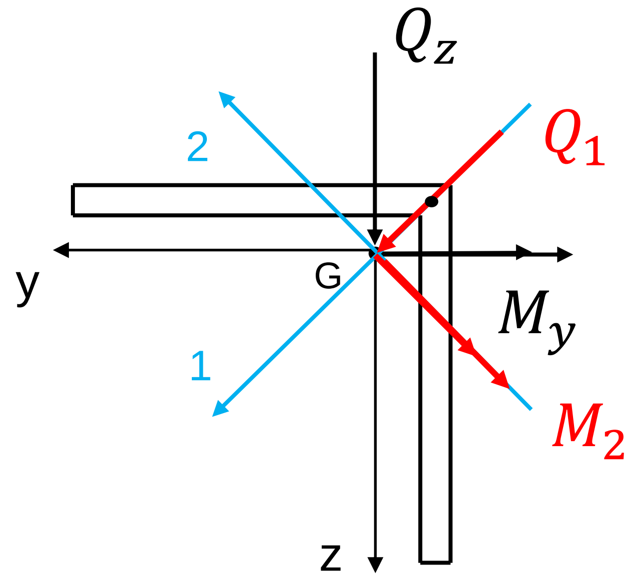
$$M_1 = M_y \cdot \cos 45^0 = M_2$$

**¡Observación!**
Aparece un efecto de torsión debido a que Q no pasa por el centro de corte

# Desplazamiento $\eta_{A,1}$

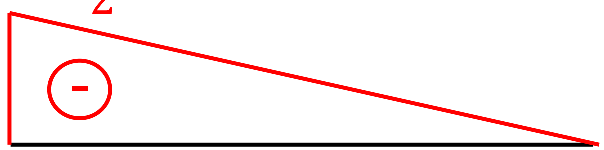
- Diagramas de la deformación virtual



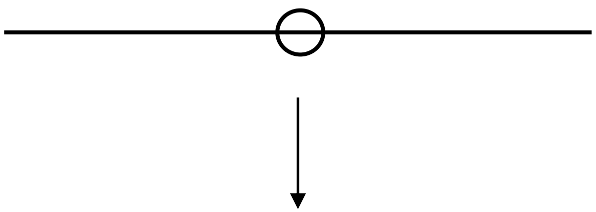




$\frac{\sqrt{2}}{2} \cdot P \cdot L$

$M_{2,P}$

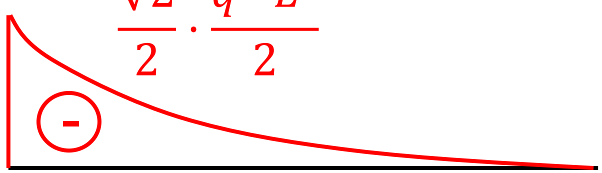



$Mt$



$\frac{\sqrt{2}}{2} \cdot \frac{q \cdot L^2}{2}$

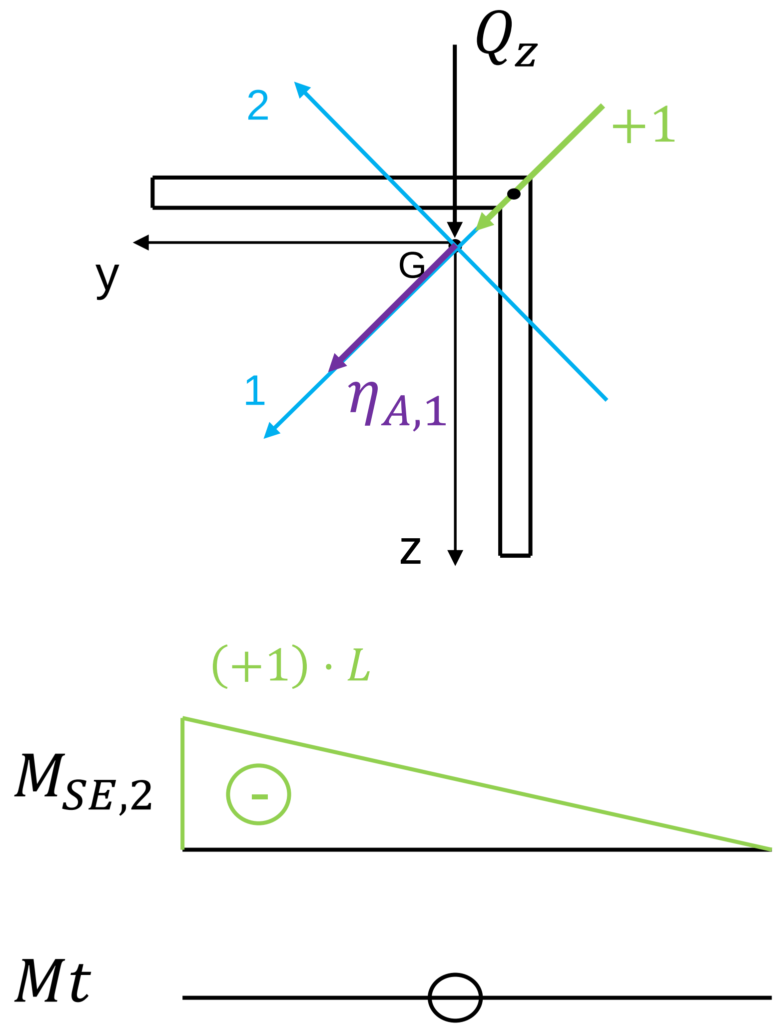
$M_{2,q}$




$Q_1$ pasa
por el centro de corte

- Sistema equilibrado ⟶ Aplicamos fuerza unitaria en **A** con dirección 1

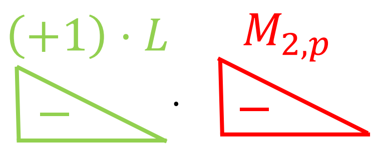
  - *Sección A*

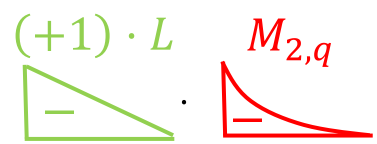


- Aplicamos el **Teorema de los trabajos virtuales**

$$(+1)\cdot\eta_{A,1}=\int\frac{M_2^{SE}\cdot M_2^{DV}}{E\cdot J_2}\mathrm{d}x+\underbrace{\int\frac{M_t^{SE}\cdot M_t^{DV}}{G\cdot J}dx}_{=0}$$

$$(+1)\cdot\eta_{A,1}=\int_0^L\frac{[(+1)\cdot L]\cdot[M_{2,p}]}{E\cdot J_2}\mathrm{d}x+\int_0^L\frac{[(+1)\cdot L]\cdot[M_{2,q}]}{E\cdot J_2}\mathrm{d}x$$



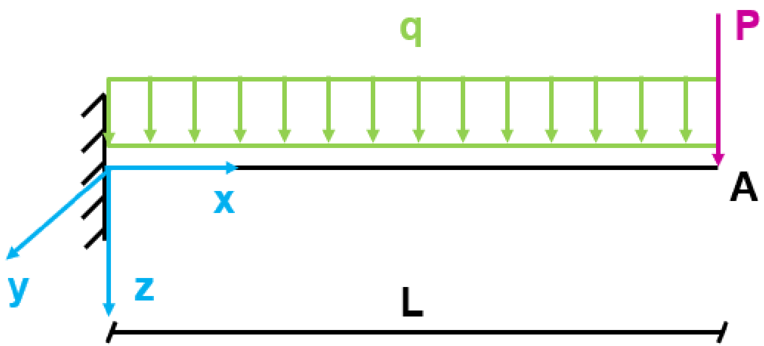



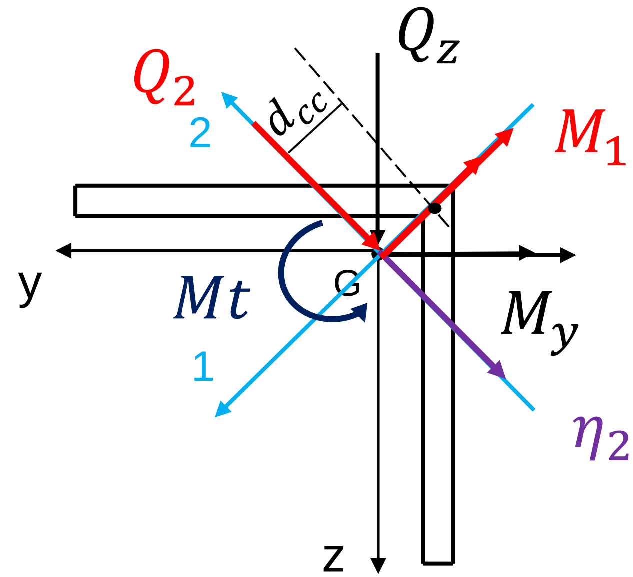

$$(+1)\cdot\eta_{A,1}=\frac{1}{3}\cdot\left(-(+1)\cdot L\right)\cdot\frac{\sqrt{2}}{2}\cdot\frac{(-P\cdot L)}{E\cdot J_2}\cdot L+\frac{1}{4}\cdot\left(-(+1)\cdot L\right)\cdot\frac{\sqrt{2}}{2}\cdot\frac{(-q\cdot L^2)}{2\cdot E\cdot J_2}\cdot L$$

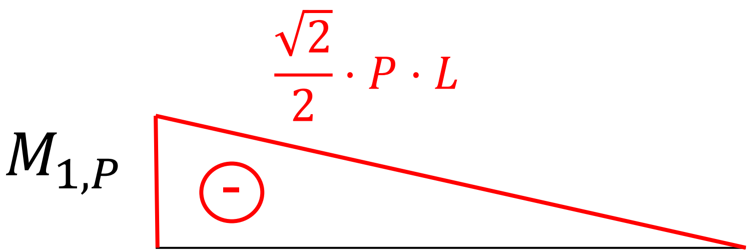
$$\boxed{\eta_{A,1}=\frac{\sqrt{2}}{2\cdot E\cdot J_2}\cdot\left(\frac{P\cdot L^3}{3}+\frac{q\cdot L^4}{8}\right)}$$

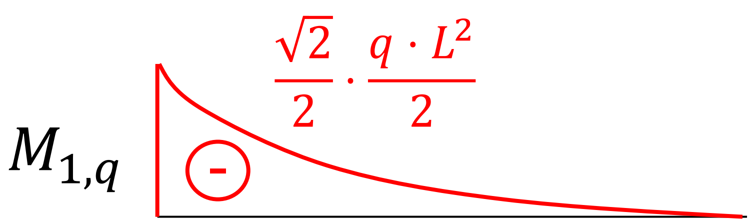
# Desplazamiento $\eta_2$

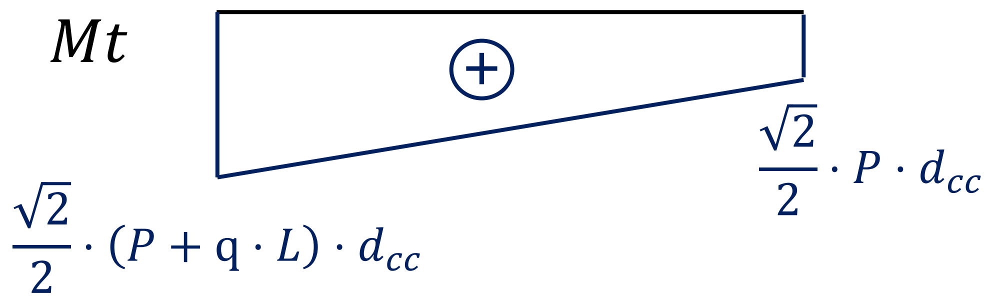
- Diagramas de deformación virtual



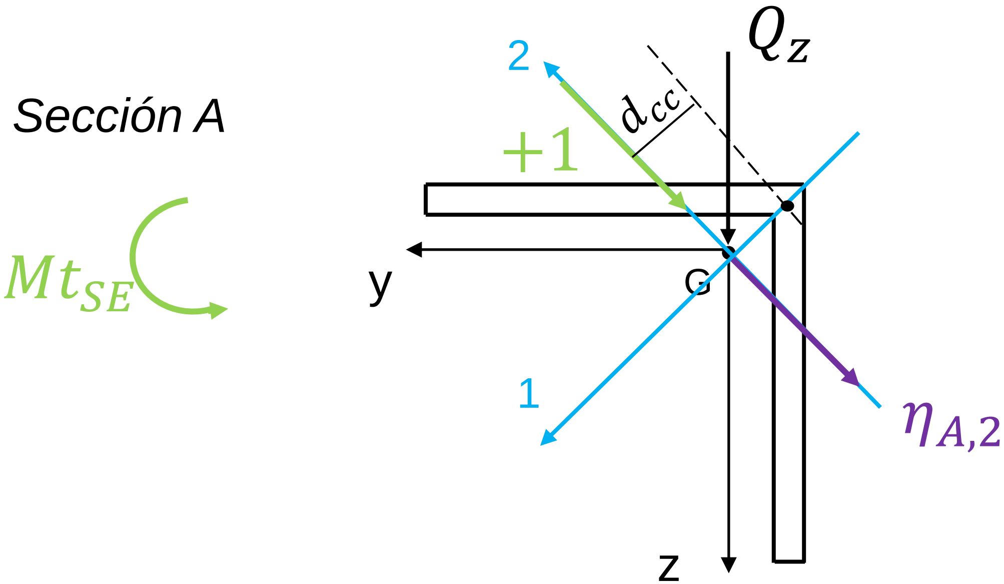



- Sistema equilibrado → Aplicamos **fuerza unitaria en A** con dirección 2

  - *Sección A*

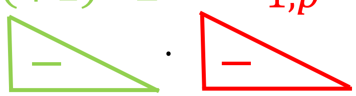


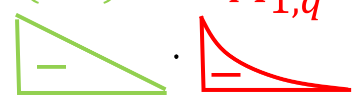


- Aplicamos el **Teorema de los trabajos virtuales**

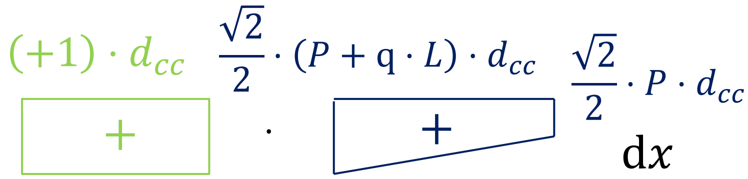
$$(+1)\cdot\eta_{A,2} = \underbrace{\int \frac{M_1^{SE}\cdot M_1^{DV}}{E\cdot J_1}\,\mathrm{d}x}_{①} + \underbrace{\int \frac{M_t^{SE}\cdot M_t^{DV}}{G\cdot J}\,dx}_{②}$$

$$① = \int_0^L \frac{(+1)\cdot L \quad\cdot\quad M_{1,p}}{E\cdot J_1}\,\mathrm{d}x + \int_0^L \frac{(+1)\cdot L \quad\cdot\quad M_{1,q}}{E\cdot J_1}\,\mathrm{d}x$$



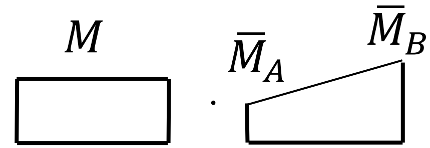




$$① = \frac{\sqrt{2}}{2\cdot E\cdot J_1}\cdot\left(\frac{P\cdot L^3}{3}+\frac{q\cdot L^4}{8}\right)$$




$$(2) = \int_0^L \frac{[\text{diagramas}]}{G\cdot J}\,dx = \frac{\sqrt{2}}{4\cdot G\cdot J}\cdot d_{cc}^2\cdot L\cdot(2\cdot P + q\cdot L)$$

Aplicando la fórmula
y reordenando




$$= \frac{1}{2}\cdot M\cdot(\bar{M}_A + \bar{M}_B)\cdot L$$

- Entonces,

$$\eta_{A,2} = (1) + (2)$$

$$\eta_{A,2} = \frac{\sqrt{2}}{2\cdot E\cdot J_1}\cdot\left(\frac{P\cdot L^3}{3} + \frac{q\cdot L^4}{8}\right) + \frac{\sqrt{2}}{4\cdot G\cdot J}\cdot d_{cc}^2\cdot L\cdot(2\cdot P + q\cdot L)$$

- Cual es el **desplazamiento máximo** del punto A?

$$\eta_{A,1} = \frac{\sqrt{2}}{2 \cdot E \cdot J_2} \cdot \left(\frac{P \cdot L^3}{3} + \frac{q \cdot L^4}{8}\right)$$

$$\eta_{A,2} = \frac{\sqrt{2}}{2 \cdot E \cdot J_1} \cdot \left(\frac{P \cdot L^3}{3} + \frac{q \cdot L^4}{8}\right) + \frac{\sqrt{2}}{4 \cdot \mathrm{G} \cdot \mathrm{J}} \cdot d_{cc}^2 \cdot L \cdot (2 \cdot P + q \cdot L)$$

- El **desplazamiento máximo** del punto A, considerando esta sección, va a ser la composición de los calculados en los ejes 1 y 2

$$\eta_A = \sqrt{\left(\eta_{A,1}\right)^2 + \left(\eta_{A,2}\right)^2}$$

**¡Observación!**
La dirección del desplazamiento no es z