FACULTAD DE INGENIERIA
Universidad de Buenos Aires

Análisis Numérico / Métodos Matemáticos y Numéricos (75.12/95.04/95.13)

Sistemas de Ecuaciones No Lineales

Introducción

SELN:

$f_1(x_1, x_2, \ldots, x_n) = 0$

$f_2(x_1, x_2, \ldots, x_n) = 0$

…

$f_n(x_1, x_2, \ldots, x_n) = 0$

nD →

$$\bar{F}(\bar{X}) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \ldots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \end{bmatrix}$$

$$\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}$$

→

<u>Métodos de res.</u>

- Newton Raphson
- Pto. Fijo
- Cuasi Newton

SENL – Ejercicio Pto Fijo

Resolver el siguiente SENL:

$$\begin{cases} {x_1}^2 + {x_2}^2 = 4 \\ x_1 x_2 = 1 \end{cases}$$

SENL – Ejercicio Pto Fijo

Resolver el siguiente SENL:

$$\begin{cases} {x_1}^2 + {x_2}^2 = 4 \\ x_1 x_2 = 1 \end{cases}$$

$$1D: \quad F(x) = 0 \quad \rightarrow \quad G(x) = x - q\,F(x) \quad \rightarrow \quad x^{k+1} = x^k - q\,F\left(x^k\right)$$

$$\text{n}D: \quad \bar{F}(\bar{X}) = 0 \quad \rightarrow \quad \bar{G}(\bar{X}) = \bar{X} - q\,\bar{F}(\bar{X}) \quad \rightarrow \quad \bar{X}^{k+1} = \bar{X}^k - q\,\bar{F}\left(\bar{X}^k\right)$$

SENL – Ejercicio Pto Fijo

Sea el siguiente SENL

$$x_1^{\,2} + x_2^{\,2} = 4 \quad \rightarrow \quad f_1(x_1, x_2) = x_1^{\,2} + x_2^{\,2} - 4 = 0$$

$$x_1 x_2 = 1 \quad \rightarrow \quad f_2(x_1, x_2) = x_1 x_2 - 1 = 0$$

$$\bar{F}(\bar{X}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_1^{\,2} + x_2^{\,2} - 4 \\ x_1 x_2 - 1 \end{bmatrix}$$

Planteo el esquema de iteración $\bar{X}^{k+1} = \bar{G}\left(\bar{X}^{k}\right)$

$$\bar{G}(\bar{X}) = \bar{X} - \bar{F}(\bar{X})$$

$$\bar{G}(\bar{X}) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1^{\,2} + x_2^{\,2} - 4 \\ x_1 x_2 - 1 \end{bmatrix}$$

$$\bar{G}(\bar{X}) = \begin{bmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_1 - x_1^{\,2} - x_2^{\,2} + 4 \\ x_2 - x_1 x_2 + 1 \end{bmatrix} \quad \Rightarrow \quad \boxed{\begin{bmatrix} x_1^{\,k+1} \\ x_2^{\,k+1} \end{bmatrix} = \begin{bmatrix} x_1^{\,k} - {x_1^{\,k}}^2 - (x_2^{\,k})^2 + 4 \\ x_2^{\,k} - x_1^{\,k} x_2^{\,k} + 1 \end{bmatrix}}$$

SENL – Ejercicio Pto Fijo

$$\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \end{bmatrix} = \begin{bmatrix} x_1^{k} - (x_1^{k})^2 - (x_2^{k})^2 + 4 \\ x_2^{k} - x_1^{k} x_2^{k} + 1 \end{bmatrix}$$

Supongamos: $\bar{X}_0 = \begin{bmatrix} 1{,}8 \\ 0{,}5 \end{bmatrix}$

$$\begin{bmatrix} x_1^{1} \\ x_2^{1} \end{bmatrix} = \begin{bmatrix} x_1^{0} - (x_1^{0})^2 - (x_2^{0})^2 + 4 \\ x_2^{0} - x_1^{0} x_2^{0} + 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1^{1} \\ x_2^{1} \end{bmatrix} = \begin{bmatrix} 1{,}8 - 1{,}8^2 - 0{,}5^2 + 4 \\ 0{,}5 - 1{,}8 * 0{,}5 + 1 \end{bmatrix} = \begin{bmatrix} 2{,}31 \\ 0{,}6 \end{bmatrix}$$

k	x_1	x_2	TOL <0,01%
0	1,8	0,5	
1	2,31	0,6	22,078%
2	0,6139	0,214	276,283%
3	4,191231	1,082625	85,353%
4	-10,5473	-2,45491	139,738%
5	-123,819	-27,3475	91,482%
6	-16198,7	-3412,47	99,236%
7	-2,7E+08	-5,5E+07	99,994%
8	-7,8E+16	-1,5E+16	100,000%
9	-6,3E+33	-1,2E+33	100,000%
10	-4,2E+67	-7,5E+66	100,000%
11	-2E+135	-3E+134	100,000%
12	-3E+270	-6E+269	100,000%

$$\text{TOL} = \frac{\|\bar{X}^{k+1} - \bar{X}^{k}\|_\infty}{\|\bar{X}^{k+1}\|_\infty}$$

$\Rightarrow$ *no converge*

SENL – Ejercicio Pto Fijo

$$f_1(x_1, x_2) = {x_1}^2 + {x_2}^2 - 4 = 0$$

$$f_2(x_1, x_2) = x_1 x_2 - 1 = 0$$

Planteo un esquema alternativo de iteración $\bar{X}^{k+1} = \bar{G}(\bar{X}^k)$

$$g_1(x_1, x_2) \Rightarrow x_1 = \sqrt{4 - {x_2}^2} \quad \rightarrow \quad {x_1}^{k+1} = \sqrt{4 - ({x_2}^k)^2}$$

$$g_2(x_1, x_2) \Rightarrow x_2 = \frac{1}{x_1} \quad \rightarrow \quad {x_2}^{k+1} = \frac{1}{{x_1}^k}$$

$$\Rightarrow \begin{bmatrix} {x_1}^{k+1} \\ {x_2}^{k+1} \end{bmatrix} = \begin{bmatrix} \sqrt{4 - ({x_2}^k)^2} \\ \frac{1}{{x_1}^k} \end{bmatrix}$$

SENL – Ejercicio Pto Fijo

Resuelvo para semilla $\bar{X}_0 = \begin{bmatrix} 1{,}8 \\ 0{,}5 \end{bmatrix}$

Simil Jacobi

$$\begin{bmatrix} {x_1}^{k+1} \\ {x_2}^{k+1} \end{bmatrix} = \begin{bmatrix} \sqrt{4-({x_2}^{k})^2} \\ \dfrac{1}{{x_1}^{k}} \end{bmatrix}$$

k	x_1	x_2	***TOL <0,01%***
0	1,8	0,5	
1	1,93649	0,55556	7,048%
2	1,92129	0,51640	2,038%
3	1,93218	0,52048	0,564%
4	1,93109	0,51755	0,152%
5	1,93188	0,51784	0,041%
6	1,93180	0,51763	0,011%
7	1,93185	0,51765	0,003%

TOL = $\dfrac{\|\bar{X}^{k+1} - \bar{X}^{k}\|_\infty}{\|\bar{X}^{k+1}\|_\infty}$

Simil Gauss Seidel

$$\begin{bmatrix} {x_1}^{k+1} \\ {x_2}^{k+1} \end{bmatrix} = \begin{bmatrix} \sqrt{4-({x_2}^{k})^2} \\ \dfrac{1}{{x_1}^{k+1}} \end{bmatrix}$$

k	x_1	x_2	TOL <0,01%
0	1,8	0,5	
1	1,93649	0,51640	7,048%
2	1,93218	0,51755	0,223%
3	1,93188	0,51763	0,016%
4	1,93185	0,51764	0,001%

SENL – Ejercicio NR

Resolver el siguiente SENL:

$$\frac{{x_1}^2}{25} = 1 - \frac{{x_2}^2}{4}$$

$$2x_1 + 5x_2 = 10$$

SENL – Ejercicio NR

Resolver el siguiente SENL:

$$\begin{cases} \frac{{x_1}^2}{25} = 1 - \frac{{x_2}^2}{4} \\ 2x_1 + 5x_2 = 10 \end{cases}$$

$$1D: \quad F(x) = 0 \quad \rightarrow \quad G(x) = x - \frac{1}{F'(x)} \; F(x) \quad \rightarrow \quad x^{k+1} = x^k - \frac{1}{F'(x^k)} \; F\left(x^k\right)$$

$$\mathrm{n}D: \quad \bar{F}(\bar{X}) = 0 \quad \rightarrow \quad \bar{G}(\bar{X}) = \bar{X} - \bar{\bar{J}}(\bar{X})^{-1} \; \bar{F}(\bar{X}) \quad \rightarrow \quad \bar{X}^{k+1} = \bar{X}^k - \bar{\bar{J}}(\bar{X})^{-1} \; \bar{F}\left(\bar{X}^k\right)$$

$$\bar{\bar{J}}(\bar{X}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$$

SENL – Ejercicio NR

Sea el siguiente SENL

$$\frac{{x_1}^2}{25} = 1 - \frac{{x_2}^2}{4} \quad \rightarrow \quad f_1(x_1, x_2) = \frac{{x_1}^2}{25} - 1 + \frac{{x_2}^2}{4} = 0$$

$$2x_1 + 5x_2 = 10 \quad \rightarrow \quad f_2(x_1, x_2) = 2x_1 + 5x_2 - 10 = 0$$

$$\bar{F}(\bar{X}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} \frac{{x_1}^2}{25} - 1 + \frac{{x_2}^2}{4} \\ 2x_1 + 5x_2 - 10 \end{bmatrix}$$

$$\bar{G}(\bar{X}) = \bar{X} - \bar{\bar{J}}(\bar{X})^{-1} \; \bar{F}(\bar{X}) \quad \rightarrow$$

esquema de iteración: $\bar{X}^{k+1} = \bar{X}^k - \bar{\bar{J}}\left(\bar{X}^k\right)^{-1} \; \bar{F}\left(\bar{X}^k\right)$

$$\begin{bmatrix} {x_1}^{k+1} \\ {x_2}^{k+1} \end{bmatrix} = \begin{bmatrix} {x_1}^{k} \\ {x_2}^{k} \end{bmatrix} - \begin{bmatrix} \frac{2}{25}{x_1}^k & \frac{1}{2}{x_2}^k \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} \frac{({x_1}^k)^2}{25} - 1 + \frac{({x_2}^k)^2}{4} \\ 2{x_1}^k + 5{x_2}^k - 10 \end{bmatrix}$$

$$\bar{\bar{J}}(\bar{X}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{2}{25}x_1 & \frac{1}{2}x_2 \\ 2 & 5 \end{bmatrix}$$

$$\bar{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

SENL – Ejercicio NR

$$\bar{X}^{k+1} = \bar{X}^{k} - \bar{\bar{J}}(\bar{X}^{k})^{-1}\ \bar{F}(\bar{X}^{k})$$

$$\begin{bmatrix} {x_1}^{k+1} \\ {x_2}^{k+1} \end{bmatrix} = \begin{bmatrix} {x_1}^{k} \\ {x_2}^{k} \end{bmatrix} - \begin{bmatrix} \frac{2}{25}{x_1}^{k} & \frac{1}{2}{x_2}^{k} \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} \frac{({x_1}^{k})^2}{25} - 1 + \frac{({x_2}^{k})^2}{4} \\ 2{x_1}^{k} + 5{x_2}^{k} - 10 \end{bmatrix}$$

k=0 →

$$\bar{X}^0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} {x_1}^{1} \\ {x_2}^{1} \end{bmatrix} = \begin{bmatrix} {x_1}^{0} \\ {x_2}^{0} \end{bmatrix} - \begin{bmatrix} \frac{2}{25}{x_1}^{0} & \frac{1}{2}{x_2}^{0} \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} \frac{({x_1}^{0})^2}{25} - 1 + \frac{({x_2}^{0})^2}{4} \\ 2{x_1}^{0} + 5{x_2}^{0} - 10 \end{bmatrix}$$

$$\begin{bmatrix} {x_1}^{1} \\ {x_2}^{1} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} \frac{4}{25} & 1 \\ 2 & 5 \end{bmatrix}^{-1} \begin{bmatrix} \frac{4}{25} - 1 + 1 \\ 4 + 10 - 10 \end{bmatrix}$$

$$\begin{bmatrix} {x_1}^{1} \\ {x_2}^{1} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 2,6666 \\ -0,2666 \end{bmatrix}$$

$$\begin{bmatrix} {x_1}^{1} \\ {x_2}^{1} \end{bmatrix} = \begin{bmatrix} -0,6666 \\ 2,2666 \end{bmatrix}$$

Tengo que calcular el Jacobiano y su inversa, en cada iteración…muy costoso computacionalmente

SENL – Ejercicio NR

Planteo

$$\bar{X}^{k+1} = \bar{X}^{k} - \bar{\bar{J}}(\bar{X}^{k})^{-1}\ \bar{F}(\bar{X}^{k})$$

$$\underbrace{\bar{X}^{k+1} - \bar{X}^{k}} = -\ \bar{\bar{J}}(\bar{X}^{k})^{-1}\ \bar{F}(\bar{X}^{k})$$

$$\Delta\bar{X}^{k+1} = -\ \bar{\bar{J}}(\bar{X}^{k})^{-1}\ \bar{F}(\bar{X}^{k})$$

$$\boxed{\bar{\bar{J}}(\bar{X}^{k})\ \ \Delta\bar{X}^{k+1} = -\ \bar{F}(\bar{X}^{k})} \quad \Rightarrow$$

SEL despejo $\Delta\bar{X}^{k+1} \rightarrow$ evito calcular la inversa del $\bar{\bar{J}}$

Tengo que resolver un SEL para cada iteración k

$$\boxed{\bar{X}^{k+1} = \bar{X}^{k} + \Delta\bar{X}^{k+1}}$$

SENL – Ejercicio NR

$$\bar{\bar{J}}(\bar{X}^k) \quad \Delta\bar{X}^{k+1} = - \ \bar{F}(\bar{X}^k)$$

$$\begin{bmatrix} \frac{2}{25}{x_1}^k & \frac{1}{2}{x_2}^k \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \Delta{x_1}^{k+1} \\ \Delta{x_2}^{k+1} \end{bmatrix} = -\begin{bmatrix} \frac{({x_1}^2)^k}{25} - 1 + \frac{({x_2}^2)^k}{4} \\ 2{x_1}^k + 5{x_2}^k - 10 \end{bmatrix}$$

k=0 $\bar{X}^0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} \frac{4}{25} & 1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} \Delta{x_1}^{k+1} \\ \Delta{x_2}^{k+1} \end{bmatrix} = - \begin{bmatrix} \frac{4}{25} \\ 4 \end{bmatrix}$$

→ SEL!

$$\begin{bmatrix} \Delta{x_1}^{k+1} \\ \Delta{x_2}^{k+1} \end{bmatrix} = \begin{bmatrix} -2,6666 \\ 0,2666 \end{bmatrix}$$

$$\Rightarrow \quad \bar{X}^1 = \bar{X}^0 + \Delta\bar{X}^1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -2,6666 \\ 0,2666 \end{bmatrix}$$

$$\bar{X}^1 = \begin{bmatrix} -0,6666 \\ 2,2666 \end{bmatrix}$$

SENL – Ejercicio NR

$\bar{X}^0 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

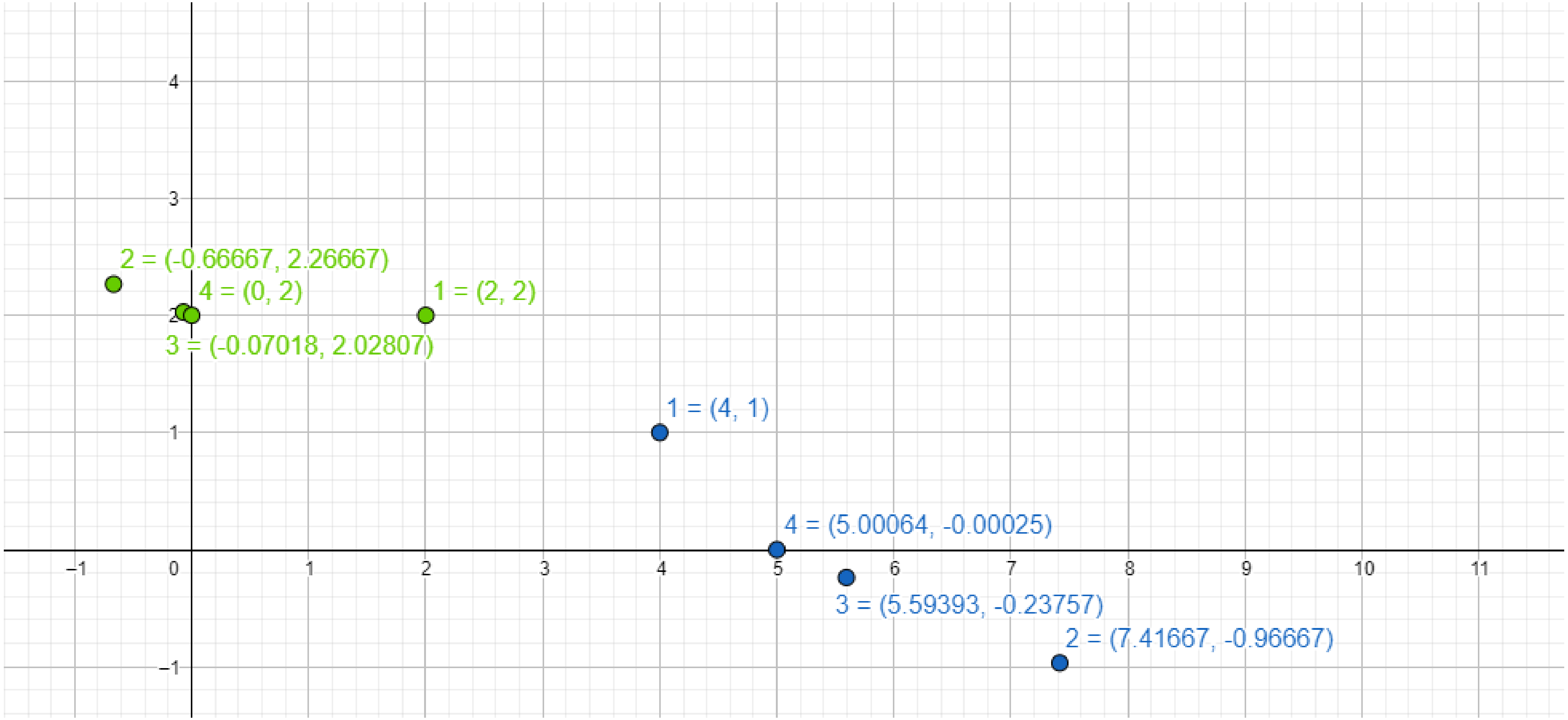
k	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{\Delta x_1 = \lvert {x_1}^{k+1} - {x_1}^{k} \rvert}$	$\boldsymbol{\Delta x_2 = \lvert {x_2}^{k+1} - {x_2}^{k} \rvert}$	P1	P2
0	2	2	-	-		
1	-0,66667	2,26667	2,66667	0,26667		
2	-0,07018	2,02807	0,59649	0,23860		
3	-0,00096	2,00038	0,06922	0,02769	1,43825	19,36435
4	0,00000	2,00000	0,00096	0,00038	1,98732	1,98732
5	-7,51E-15	2	1,83E-07	7,34E-08	1,99991	1,99991
6	-9,42E-16	2	6,57E-15	2,66E-15	2,00294	2,00127

SENL – Ejercicio NR

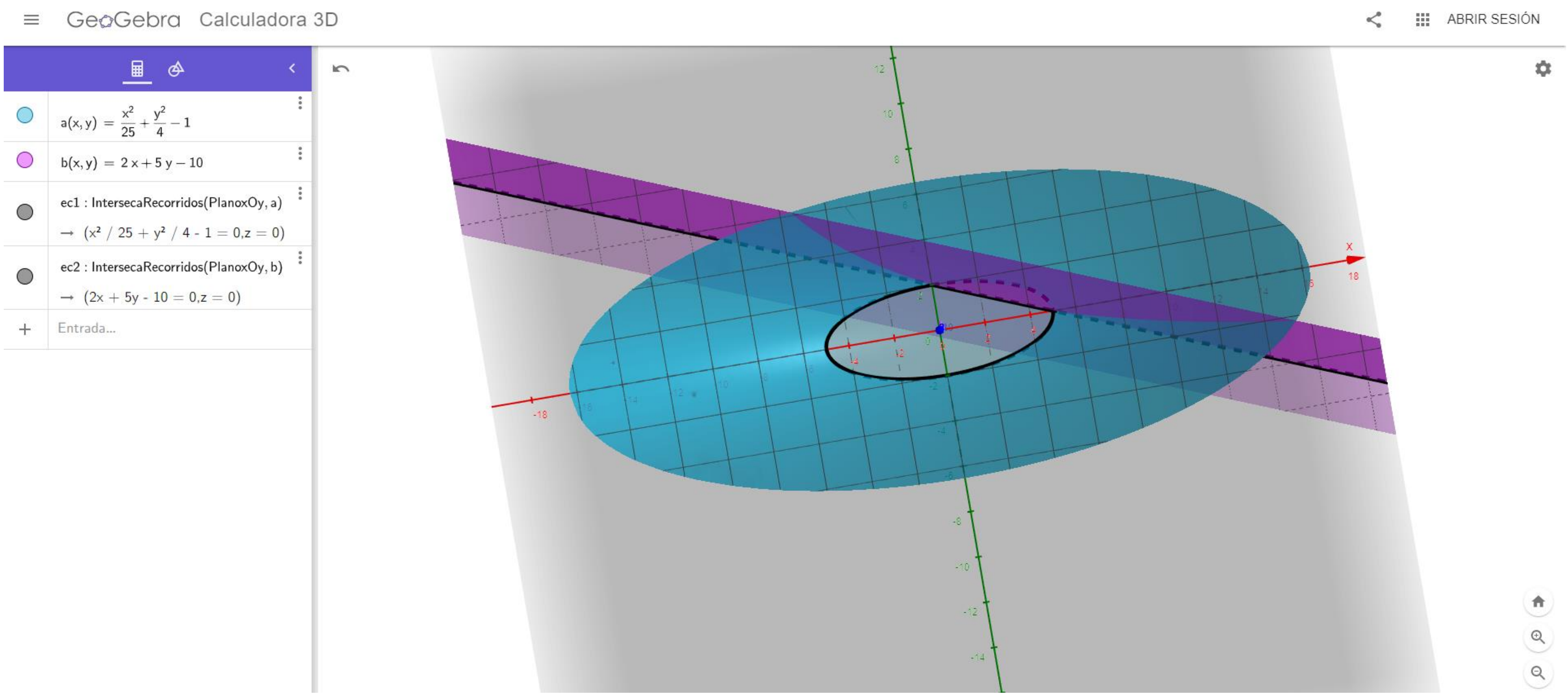
$$\bar{X}^0 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

k	x_1	x_2	$\Delta x_1 = \lvert x_1^{k+1} - x_1^k \rvert$	$\Delta x_1 = \lvert x_1^{k+1} - x_1^k \rvert$	P1	P2
0	4	1				
1	7,41667	-0,96667	5,41667	2,96667		
2	5,59393	-0,23757	1,82274	0,72910		
3	5,05701	-0,02280	0,53692	0,21477	1,12221	0,87093
4	5,00064	-0,00025	0,05637	0,02255	1,84405	1,84405
5	5,00E+00	-3,23E-08	6,35E-04	2,54E-04	1,99011	1,99011
6	5,00E+00	-5,52E-16	8,07E-08	3,23E-08	1,99994	1,99994

SENL – Ejercicio NR



SENL – Ejercicio NR



SENL – Método Cuasi Newton

En Newton Raphson: $\bar{X}^{k+1} = \bar{X}^k - \bar{\bar{J}}(\bar{X}^k)^{-1} \bar{F}(\bar{X}^k)$ ⇒ $\bar{\bar{J}}(\bar{X}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$ Derivadas analíticas

En Cuasi Newton: reemplazo con aproximaciones discretas de las derivadas parciales

$$\frac{\partial \bar{F}_i(\bar{X}_k)}{\partial x_j} \approx \frac{\bar{F}_i(\bar{X}_k + he_j) - \bar{F}_i(\bar{X}_k)}{h}$$

⇒ Matriz Jacobiana aproximada

$$f_1(x_1, x_2) = \frac{{x_1}^2}{25} - 1 + \frac{{x_2}^2}{4} \quad \Rightarrow \quad \frac{\partial f_1}{\partial x_1} \approx \frac{\left[\frac{(x_1 + h)^2}{25} - 1 + \frac{{x_2}^2}{4}\right] - \left[\frac{{x_1}^2}{25} - 1 + \frac{{x_2}^2}{4}\right]}{h} \qquad \frac{\partial f_1}{\partial x_2} \approx \frac{\left[\frac{{x_1}^2}{25} - 1 + \frac{(x_2 + h)^2}{4}\right] - \left[\frac{{x_1}^2}{25} - 1 + \frac{{x_2}^2}{4}\right]}{h}$$

$$f_2(x_1, x_2) = 2x_1 + 5x_2 - 10 \quad \Rightarrow \quad \frac{\partial f_2}{\partial x_1} \approx \frac{[2(x_1 + h) + 5x_2 - 10] - [2x_1 + 5x_2 - 10]}{h} \qquad \frac{\partial f_2}{\partial x_2} \approx \frac{[2x_1 + 5(x_2 + h) - 10] - [2x_1 + 5x_2 - 10]}{h}$$

SENL – Método Cuasi Newton

En Newton Raphson: $\bar{X}^{k+1} = \bar{X}^{k} - \bar{\bar{J}}(\bar{X}^{k})^{-1}\ \bar{F}(\bar{X}^{k})$ ⇒ $\bar{\bar{J}}(\bar{X}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$ Derivadas analíticas

¿Qué ocurre si no actualizo la matriz jacobiana en cada iteración?

Calculo $\bar{\bar{J}}_1$ ⇒ "n" iteraciones

$$\bar{\bar{J}}_1\ \Delta\bar{X}^1 = -\ \bar{F}(\bar{X}^1)$$

$$\bar{\bar{J}}_1\ \Delta\bar{X}^2 = -\ \bar{F}(\bar{X}^2)$$

...

$$\bar{\bar{J}}_1\ \Delta\bar{X}^n = -\ \bar{F}(\bar{X}^n)$$

⇒ Actualizo $\bar{\bar{J}}_2$ ⇒ "n" iteraciones

$$\bar{\bar{J}}_2\ \Delta\bar{X}^1 = -\ \bar{F}(\bar{X}^{1*})$$

$$\bar{\bar{J}}_2\ \Delta\bar{X}^2 = -\ \bar{F}(\bar{X}^{2*})$$

...

$$\bar{\bar{J}}_2\ \Delta\bar{X}^n = -\ \bar{F}(\bar{X}^{n*})$$