Sistemas de Ec. Lineales

-Métodos Iterativos-

Eje. Resolver el siguiente sistema utilizando el método de **Jacobi**, **Gauss Seidel** y **SOR** (w=1,1), hasta obtener una tolerancia<0,1%. Trabajar con 5 dígitos de precisión.

$$\begin{aligned} 4x + 3y \quad &= 24 \\ 3x + 4y - z &= 30 \\ -y + 4z &= -24 \end{aligned}$$

JACOBI

$$x^{(k+1)} = \frac{24 - 3y^{(k)}}{4}$$

$$y^{(k+1)} = \frac{30 - 3x^{(k)} + z^{(k)}}{4}$$

$$z^{(k+1)} = \frac{-24 + y^{(k)}}{4}$$

GAUSS SEIDEL

$$x^{(k+1)} = \frac{24 - 3y^{(k)}}{4}$$

$$y^{(k+1)} = \frac{30 - 3x^{(k+1)} + z^{(k)}}{4}$$

$$z^{(k+1)} = \frac{-24 + y^{(k+1)}}{4}$$

SOR

$$x^{(k+1)} = \left(\frac{24 - 3y^{(k)}}{4} - x^{(k)}\right) * w + x^{(k)}$$

$$y^{(k+1)} = \left(\frac{30 - 3x^{(k+1)} + z^{(k)}}{4} - y^{(k)}\right) * w + y^{(k)}$$

$$z^{(k+1)} = \left(\frac{-24 + y^{(k+1)}}{4} - z^{(k)}\right) * w + z^{(k)}$$

$$\bar{R}_{GS} = \bar{X}^{(k+1)} - \bar{X}^{(k)}$$

JACOBI

$$x^{(k+1)} = \frac{24 - 3y^{(k)}}{4}$$
$$y^{(k+1)} = \frac{30 - 3x^{(k)} + z^{(k)}}{4}$$
$$z^{(k+1)} = \frac{-24 + y^{(k)}}{4}$$

Para un semilla arbitraria $(x^0, y^0, z^0) = (0,0,0)$

→Primera iteración $\boldsymbol{k = 0}$

$$x^{(1)} = \frac{24 - 3y^{(0)}}{4} \quad\rightarrow\quad x^{(1)} = \frac{24 - 3 * 0}{4} = 6{,}0000$$

$$y^{(1)} = \frac{30 - 3x^{(0)} + z^{(0)}}{4} \quad\rightarrow\quad y^{(1)} = \frac{30 - 3 * 0 + 0}{4} = 7{,}5000$$

$$z^{(1)} = \frac{-24 + y^{(0)}}{4} \quad\rightarrow\quad z^{(1)} = \frac{-24 + 0}{4} = -6{,}0000$$

TABLA DE ITERACIÓN

k	x	y	z	TOL(<0,1%)
0	0,0000	0,0000	0,0000	
1	6,0000	7,5000	-6,0000	100,00%
2	0,37500	1,5000	-4,1250	400,00%
3	4,8750	6,1875	-5,6250	75,76%
4	1,3594	2,4375	-4,4531	153,85%
5	4,1719	5,3672	-5,3906	54,59%
6	1,9746	3,0234	-4,6582	77,52%
7	3,7324	4,8545	-5,2441	37,72%
8	2,3591	3,3896	-4,7864	43,22%
9	3,4578	4,5341	-5,1526	25,24%
10	2,5995	3,6185	-4,8665	25,30%
11	3,2861	4,3338	-5,0954	16,50%
12	2,7497	3,7616	-4,9166	15,21%
13	3,1788	4,2086	-5,0596	10,62%
14	2,8435	3,8510	-4,9478	9,29%
15	3,1118	4,1304	-5,0373	6,76%
16	2,9022	3,9069	-4,9674	5,72%
17	3,0698	4,0815	-5,0233	4,28%
18	2,9389	3,9418	-4,9796	3,54%
19	3,0437	4,0509	-5,0146	2,69%
20	2,9618	3,9636	-4,9873	2,20%
21	3,0273	4,0318	-5,0091	1,69%
22	2,9761	3,9773	-4,9920	1,37%
23	3,0171	4,0199	-5,0057	1,06%
24	2,9851	3,9858	-4,9950	0,86%
25	3,0107	4,0124	-5,0036	0,66%
26	2,9907	3,9911	-4,9969	0,53%
27	3,0067	4,0078	-5,0022	0,42%
28	2,9942	3,9944	-4,9981	0,33%
29	3,0042	4,0049	-5,0014	0,26%
30	2,9964	3,9965	-4,9988	0,21%
31	3,0026	4,0030	-5,0009	0,16%
32	2,9977	3,9978	-4,9992	0,13%
33	3,0016	4,0019	-5,0005	0,10%
34	2,9986	3,9986	-4,9995	0,08%

$$TOL = \frac{||\bar{x}^{k+1} - \bar{x}^{k}||_\infty}{||\bar{x}^{k+1}||_\infty}$$

GAUSS SEIDEL

$$x^{(k+1)} = \frac{24 - 3y^{(k)}}{4}$$

$$y^{(k+1)} = \frac{30 - 3x^{(k+1)} + z^{(k)}}{4}$$

$$z^{(k+1)} = \frac{-24 + y^{(k+1)}}{4}$$

Para un semilla arbitraria $(x^0, y^0, z^0) = (0,0,0)$

→Primera iteración $\boldsymbol{k = 0}$

$$x^{(1)} = \frac{24 - 3y^{(0)}}{4}$$

$$y^{(1)} = \frac{30 - 3x^{(1)} + z^{(0)}}{4}$$

$$z^{(1)} = \frac{-24 + y^{(1)}}{4}$$

→

$$x^{(1)} = \frac{24 - 3 * 0}{4} = 6{,}0000$$

$$y^{(1)} = \frac{30 - 3 * 6{,}0000 + 0}{4} = 3{,}0000$$

$$z^{(1)} = \frac{-24 + 3{,}0000}{4} = -5{,}2500$$

TABLA DE ITERACIÓN

k	x	y	z	TOL(<0,1%)
0	0,0000	0,0000	0,0000	
1	6,0000	3,0000	-5,2500	100,00%
2	3,7500	3,3750	-5,1563	60,00%
3	3,4688	3,6094	-5,0977	7,79%
4	3,2930	3,7559	-5,0610	4,68%
5	3,1831	3,8474	-5,0381	2,86%
6	3,1144	3,9046	-5,0238	1,76%
7	3,0715	3,9404	-5,0149	1,09%
8	3,0447	3,9627	-5,0093	0,68%
9	3,0279	3,9767	-5,0058	0,42%
10	3,0175	3,9854	-5,0036	0,26%
11	3,0109	3,9909	-5,0023	0,16%
12	3,0068	3,9943	-5,0014	0,10%
13	**3,0043**	**3,9964**	**-5,0009**	0,06%

$$TOL = \frac{||\bar{x}^{k+1} - \bar{x}^{k}||_\infty}{||\bar{x}^{k+1}||_\infty}$$

SOR

$$x^{(k+1)} = \left(\frac{24 - 3y^{(k)}}{4} - x^{(k)}\right) * w + x^{(k)}$$

$$y^{(k+1)} = \left(\frac{30 - 3x^{(k+1)} + z^{(k)}}{4} - y^{(k)}\right) * w + y^{(k)}$$

$$z^{(k+1)} = \left(\frac{-24 + y^{(k+1)}}{4} - z^{(k)}\right) * w + z^{(k)}$$

TABLA	w= 1,1000			
k	x	y	z	TOL(<0,1%)
0	0,0000	0,0000	0,0000	
1	6,6000	2,8050	-5,8286	100,000%
2	3,6259	3,3753	-5,0889	82,025%
3	3,4528	3,6645	-5,0834	7,891%
4	3,2315	3,8196	-5,0413	5,793%
5	3,1257	3,9030	-5,0225	2,713%
6	3,0675	3,9479	-5,0121	1,475%
7	3,0363	3,9720	-5,0065	0,785%
8	3,0195	3,9849	-5,0035	0,421%
9	3,0105	3,9919	-5,0019	0,226%
10	3,0056	3,9956	-5,0010	0,121%
11	**3,0030**	**3,9977**	**-5,0005**	0,07%

$$TOL = \frac{||\bar{x}^{k+1} - \bar{x}^{k}||_\infty}{||\bar{x}^{k+1}||_\infty}$$

Para un semilla arbitraria $(x^0, y^0, z^0) = (0,0,0)$ y con w=1,1

→Primera iteración $\boldsymbol{k = 0}$

$$x^{(1)} = \left(\frac{24 - 3y^{(0)}}{4} - x^{(0)}\right) * 1,1 + x^{(0)}$$

$$y^{(1)} = \left(\frac{30 - 3x^{(1)} + z^{(0)}}{4} - y^{(0)}\right) * 1,1 + y^{(0)}$$

$$z^{(1)} = \left(\frac{-24 + y^{(1)}}{4} - z^{(0)}\right) * 1,1 + z^{(0)}$$

→

$$x^{(1)} = \left(\frac{24 - 3 * 0}{4} - 0\right) * 1,1 + 0 = 6,6000$$

$$y^{(1)} = \left(\frac{30 - 3 * 6,6000 + 0}{4} - 0\right) * 1,1 + 0 = 2,8050$$

$$z^{(1)} = \left(\frac{-24 + 2,8050}{4} - 0\right) * 1,1 + 0 = -5,8286$$

Forma indicial

JACOBI

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j\neq i}}^{n} \left(- a_{ij} x_j^{(k-1)} \right) + b_i \right]$$

GAUSS SEIDEL

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[- \sum_{j=1}^{i-1} \left(a_{ij} x_j^{(k)} \right) - \sum_{j=i+1}^{n} \left(a_{ij} x_j^{(k-1)} \right) + b_i \right]$$

SOR

$$x_i^{(k)} = (1-\omega) x_i^{(k-1)} + \frac{\omega}{a_{ii}} \left[b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)} \right]$$

Forma matricial

$$\bar{\bar{A}} \,.\, \bar{x} = \bar{b}$$

$$\downarrow$$

$$\boxed{\bar{x}^{(k+1)} = \bar{\bar{T}} \,.\, \bar{x}^{(k)} + \bar{c}}$$

$$\bar{\bar{A}}.\bar{x} = \bar{b} \qquad \bar{\bar{A}} = \bar{\bar{D}} - \bar{\bar{L}} - \bar{\bar{U}}$$

Forma matricial $\bar{x}^{(k+1)} = \bar{\bar{T}}.\bar{x}^{(k)} + \bar{c}$

JACOBI

$$x^{(k+1)} = \frac{24 - 3y^{(k)}}{4}$$

$$y^{(k+1)} = \frac{30 - 3x^{(k)} + z^{(k)}}{4}$$

$$z^{(k+1)} = \frac{-24 + y^{(k)}}{4}$$

$$\rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix}^{(k+1)} = \underbrace{\begin{bmatrix} 0 & -3/4 & 0 \\ -3/4 & 0 & 1/4 \\ 0 & 1/4 & 0 \end{bmatrix}}_{\downarrow} . \begin{bmatrix} x \\ y \\ z \end{bmatrix}^{(k)} + \begin{bmatrix} 24/4 \\ 30/4 \\ -24/4 \end{bmatrix}$$

$$\boxed{\bar{\bar{T}}_J = \bar{\bar{D}}^{-1}(\bar{\bar{L}} + \bar{\bar{U}})}$$

$$\boxed{\bar{c}_J = \bar{\bar{D}}^{-1}b}$$

GAUSS SEIDEL

$$\boxed{\bar{\bar{T}}_{GS} = (\bar{\bar{D}} - \bar{\bar{L}})^{-1}\bar{\bar{U}}}$$

$$\boxed{\bar{c}_{GS} = (\bar{\bar{D}} - \bar{\bar{L}})^{-1}.b}$$

SOR

$$\boxed{\bar{\bar{T}}_{Sor} = (\bar{\bar{D}} - w\bar{\bar{L}})^{-1}.[(1-w)\bar{\bar{D}} + w\bar{\bar{U}}}$$

$$\boxed{\bar{c}_{Sor} = w(\bar{\bar{D}} - w\bar{\bar{L}})^{-1}.b}$$

$\bar{\bar{A}}\,.\,\bar{x} = \bar{b}$, $\bar{\bar{A}} = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ $\bar{x}^{(k+1)} = \bar{\bar{T}}\,.\,\bar{x}^{(k)} + \bar{c}$

Convergencia

Teo 1) Si $\underline{\underline{A}}$ es estrictamente diagonal domin ($|a_{ii}| > \sum_{\substack{j=1 \\ i\neq j}}^{n} |a_{ij}|$) => Jacobi y Gauss-Seidel convergen

Teo 2) Si además $\underline{\underline{A}}$ es definida positiva (subdet>0) y $0<w<2$ => SOR converge

$det \begin{vmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{vmatrix} = 20$; $det \begin{vmatrix} 4 & 3 \\ 3 & 4 \end{vmatrix} = 7$

Teo 3) Si $\exists \left\| \underline{\underline{T}} \right\| < 1$ => el método converge

Teo 4) Si $\rho\left(\underline{\underline{T}}\right) = \max|\lambda_i| < 1$ <=> el método converge

Teo 5) Si $\underline{\underline{A}}$ es simétrica, def posit, tridiag en bloques => $w_{óptimo} = \dfrac{2}{1+\sqrt{1-\rho\left(\underline{\underline{T}}_{GS}\right)}}$ $= \dfrac{2}{1+\sqrt{1-\rho(\bar{\bar{T}}_{GS})}} \approx 1{,}24$

Teo 6) $\left|\underline{x}^{(k+1)} - \underline{x}\right| \leq factor * \left|\underline{x}^{(k+1)} - \underline{x}^{(k)}\right|$ cota del error de truncamiento

Convergencia

$$\begin{aligned} 4x + 3y \quad\quad &= 24 \\ 3x + 4y - z &= 30 \\ -y + 4z &= -24 \end{aligned}$$

$$\bar{\bar{A}} = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}$$

Normas

$$\bar{\bar{T}}_J = \begin{bmatrix} 0 & -3/4 & 0 \\ -3/4 & 0 & 1/4 \\ 0 & 1/4 & 0 \end{bmatrix}$$

$\left\|\bar{\bar{T}}_J\right\|_1 = 1$ $\quad\left\|\bar{\bar{T}}_{GS}\right\|_1 = 1{,}45$ $\quad\left\|\bar{\bar{T}}_{SOR}\right\|_1 = 1{,}24$

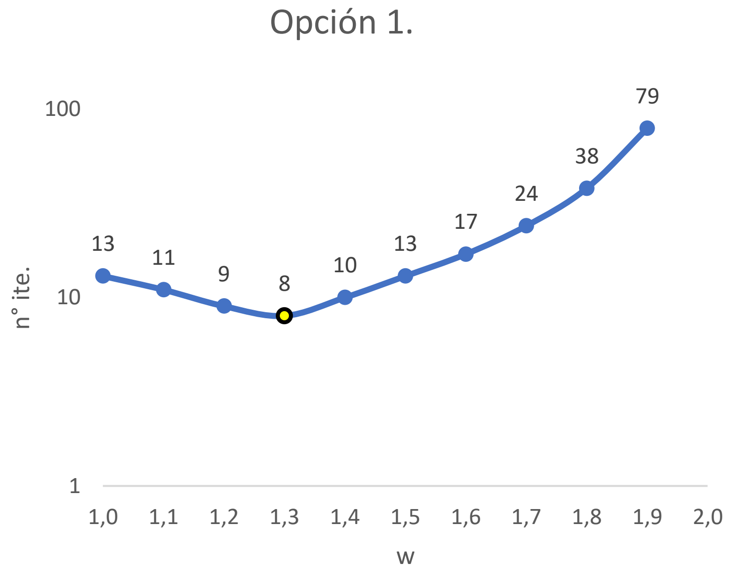
$\left\|\bar{\bar{T}}_J\right\|_\infty = 1$ $\quad\left\|\bar{\bar{T}}_{GS}\right\|_\infty = 0{,}81$ $\quad\left\|\bar{\bar{T}}_{SOR}\right\|_\infty = 0{,}93$

Radio espectral (mide la velocidad de convergencia)

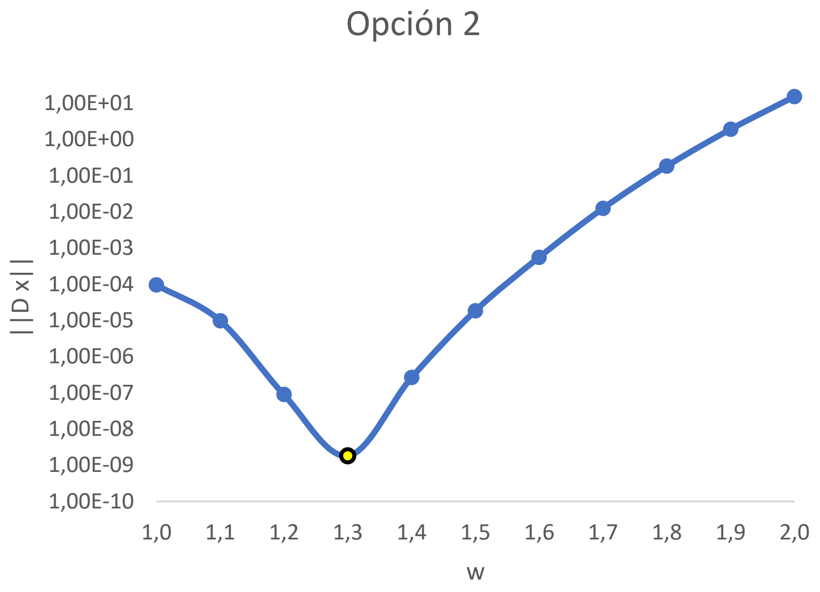
$\rho(\bar{\bar{T}}_J) = 0{,}790$ $\quad\rho(\bar{\bar{T}}_{GS}) = 0{,}625$ $\quad\rho(\bar{\bar{T}}_{SOR}) = 0{,}538$

Determinación experimental de w óptimo

Opción 1: fijar tolerancia poco restrictiva, graficar N° ite = $f(w)$
Opción 2: fijar N° ite, graficar cota del error = $f(w)$



n° ite.	w
13	1,0
11	1,1
9	1,2
8	1,3
10	1,4
13	1,5
17	1,6
24	1,7
38	1,8
79	1,9



|| Delta X ||	w
9,53E-05	1,0
9,79E-06	1,1
8,91E-08	1,2
1,82E-09	1,3
2,61E-07	1,4
1,84E-05	1,5
5,52E-04	1,6
1,23E-02	1,7
1,81E-01	1,8
1,89E+00	1,9

Orden de convergencia

Def) si $\lim_{k\to\infty} \frac{\varepsilon^{(k+1)}}{\varepsilon^{(k)p}} = \lim_{k\to\infty} \frac{\left|\underline{x}^{(k+1)} - \underline{x}\right|}{\left|\underline{x}^{(k)} - \underline{x}\right|^p} = \lambda$, entonces llamamos λ : constante asintótica del error

p : orden de convergencia

Interpretación: si $p = 1$ y $\lambda = 0.1$, el error se reduce un 90% entre cada iteración

Cómo calcular p y λ:

Como no conocemos $\underline{x}$, en lugar del error $\varepsilon^{(k+1)} = \left|\underline{x}^{(k+1)} - \underline{x}\right|$ usamos la diferencia entre las dos últimas iteraciones $\Delta x^{(k+1)} = \left|\underline{x}^{(k+1)} - \underline{x}^{(k)}\right|$, y decimos lo mismo:

$$\frac{\Delta x^{(k+1)}}{\Delta x^{(k)p}} = \frac{\left|\underline{x}^{(k+1)} - \underline{x}^{(k)}\right|}{\left|\underline{x}^{(k)} - \underline{x}^{(k-1)}\right|^p} = \lambda \text{ (en escala log es una recta)}$$

Tenemos 1 ec. con 2 inc, pero como p y λ no cambian durante todo el cálculo, planteamos la misma ecuación para 2 iteraciones sucesivas. Así:

$$\frac{\Delta x^{(k+1)}}{\Delta x^{(k)p}} = \lambda \ , \ \frac{\Delta x^{(k)}}{\Delta x^{(k-1)p}} = \lambda \text{ , entonces: } \Delta x^{(k+1)} \Delta x^{(k-1)p} = \Delta x^{(k)} \Delta x^{(k)p}$$

Despejando: $p = \dfrac{\ln\left(\Delta x^{(k+1)} / \Delta x^{(k)}\right)}{\ln\left(\Delta x^{(k)} / \Delta x^{(k-1)}\right)}$ (el método tiene que estar convirgiendo)

Gráfico 1

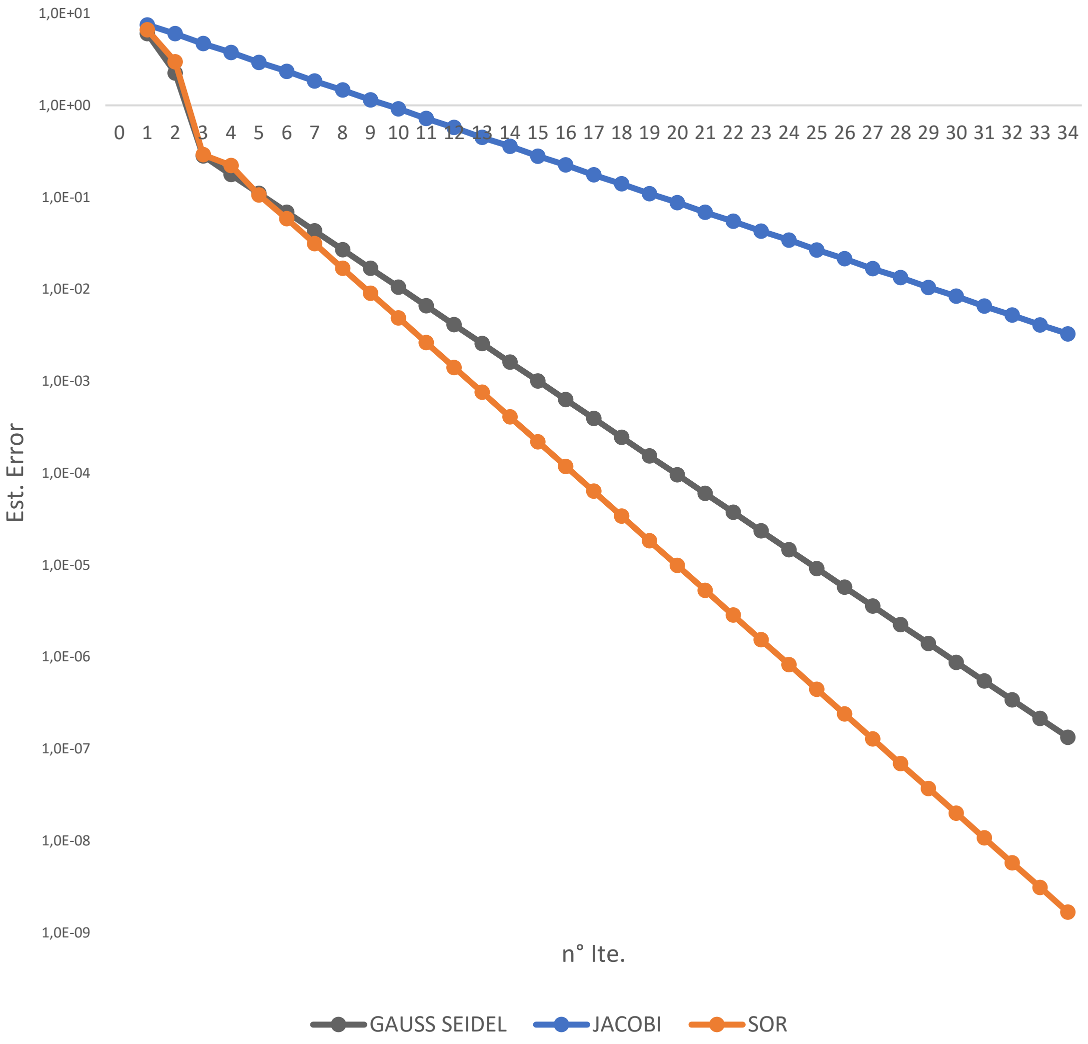
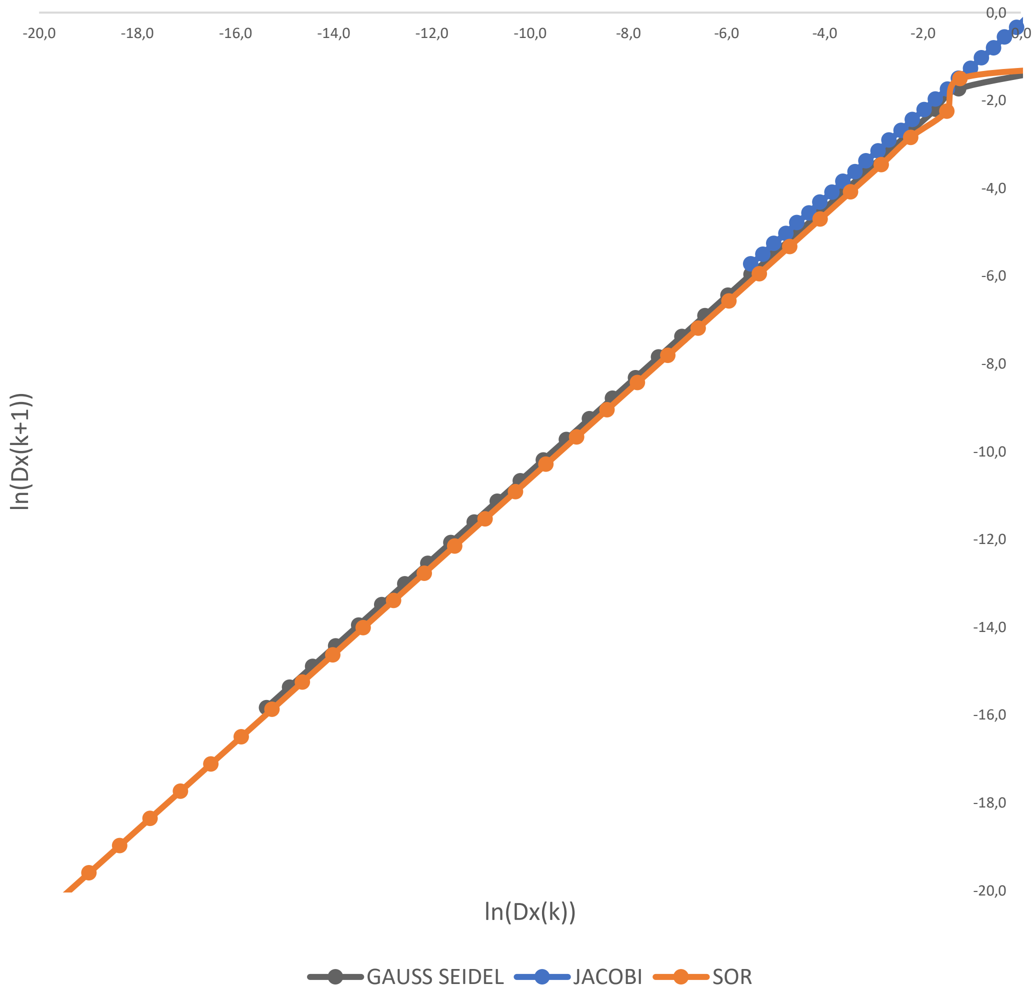


Gráfico 2

$p_J = 0{,}902$ $p_{GS} = 1{,}00$ $p_{SOR} = 1{,}00005$



$$\ln\left(D\bar{X}^{k+1}\right) = \ln(\lambda) + p\,\ln\left(D\bar{X}^{k}\right)$$

Gracias!