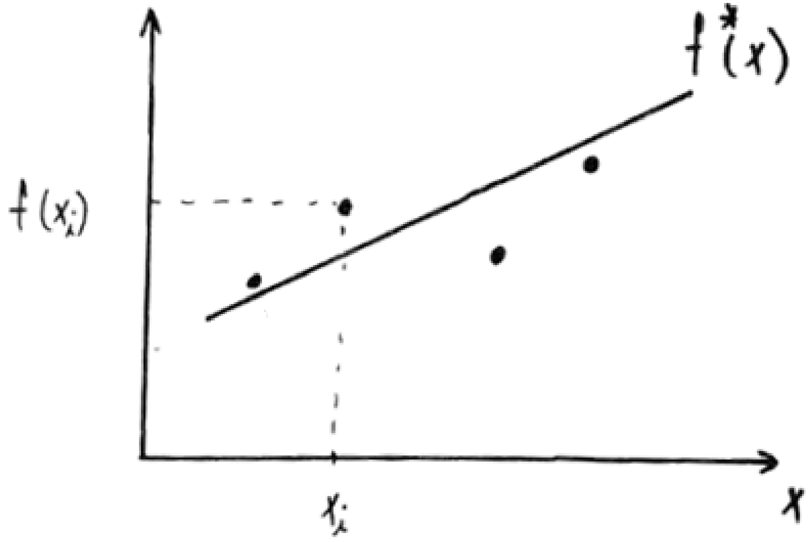
Aproximación de funciones
-Ajuste-



$m > n$

$m+1$ datos

$n+1$ incógnitas

Función aproximante $\longrightarrow$ $f^*(x_i) = \sum_{j=0}^{n} c_j \varphi_j(x_i)$

Mínimos cuadrados $\longrightarrow$ $e = \sum_{i=0}^{m} e_i^2 = \sum_{i=0}^{m} \left(f(x_i) - f^*(x_i) \right)^2$

$\downarrow$

$$\frac{\partial e}{\partial c_k} = 0 \qquad k = 0, \dots, n$$

$\downarrow$

Ec. Normales $\longrightarrow$ $\sum_{j=0}^{n} c_j \langle \varphi_j , \varphi_k \rangle = \langle f , \varphi_k \rangle \qquad k = 0, \dots, n$

Eje2) El nivel de agua en el Mar del Norte está determinado principalmente por la marea llamada M2, cuyo período es de aprox. 12 horas. Se han realizado las siguientes mediciones

t(horas)	0	2	4	6	8	10
H(t)(m)	1.0	1.6	1.4	0.6	0.2	0.8

a) Ajustar la serie de mediciones usando el método de los cuadrados mínimos y la función $H1 * (t) = ho + a1 \, . \sin\left(\frac{2\pi t}{12}\right)$

b) Calcular errores que permitan estimar la precisión de la aproximación realizada en a).

c) Utilizar ahora la función $H2 * (t) = ho + a1 \, . \sin\left(\frac{2\pi t}{12}\right) + a2 \, . \cos\left(\frac{2\pi t}{12}\right)$ y repetir b).

$$H_1^*(t) = \overbrace{ho}^{c_0} \underbrace{1}_{\varphi_0} + \overbrace{a_1}^{c_1} \underbrace{\operatorname{sen}\left(\frac{2\pi}{12} t\right)}_{\varphi_1}$$

$m = 1 \rightarrow m+1$ incógnitas $[c_0, c_1]$

$m = 5 \rightarrow m+1$ datos $H(t_i)$ en t_i

Ec. Normales $\quad \sum_{j=0}^{1} c_j < \varphi_j , \varphi_k > = < H, \varphi_k > \qquad k = 0; 1$

$k = 0 \qquad c_0 < \varphi_0, \varphi_0 > + c_1 < \varphi_1, \varphi_0 > = < H, \varphi_0 >$

$k = 1 \qquad c_0 < \varphi_0, \varphi_1 > + c_1 < \varphi_1, \varphi_1 > = < H, \varphi_1 >$

$$\Rightarrow \begin{pmatrix} \langle \varphi_0, \varphi_0 \rangle & \langle \varphi_1, \varphi_0 \rangle \\ \langle \varphi_0, \varphi_1 \rangle & \langle \varphi_1, \varphi_1 \rangle \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \langle H \varphi_0 \rangle \\ \langle H \varphi_1 \rangle \end{pmatrix}$$

$$\varphi_0(t_i) = 1 \quad \rightarrow \quad \varphi_0 = (1; 1; 1; 1; 1; 1)$$

$$\varphi_1(t_i) = \operatorname{sen}\left(\frac{2\pi}{12} t_i\right) \quad \rightarrow \quad \varphi_1 = \left(0; \frac{\sqrt{3}}{2}; \frac{\sqrt{3}}{2}; 0; -\frac{\sqrt{3}}{2}; -\frac{\sqrt{3}}{2}\right)$$

$$H(t_i) = \text{dato} \quad \rightarrow \quad H = (1; 1{,}6; 1{,}4; 0{,}6; 0{,}2; 0{,}8)$$

t(horas)	0	2	4	6	8	10
H(t)(m)	1.0	1.6	1.4	0.6	0.2	0.8

$$\langle \varphi_0, \varphi_0 \rangle = 6$$

$$\langle \varphi_1, \varphi_0 \rangle = 0$$

$$\langle \varphi_0, \varphi_1 \rangle = 0$$

$$\langle \varphi_1, \varphi_1 \rangle = 3$$

$$\langle H \varphi_0 \rangle = 5{,}6$$

$$\langle H \varphi_1 \rangle = \sqrt{3}$$

$$\rightarrow \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 5{,}6 \\ \sqrt{3} \end{pmatrix}$$

$$c_0 = h_0 = 0{,}933$$

$$c_1 = a_1 = 0{,}577$$

$$\boxed{H^*(t) = 0{,}933 + 0{,}577 \operatorname{sen}\left(\frac{2\pi}{12} t\right)}$$

b) Calcular errores que permitan estimar la precisión de la aproximación realizada en a).

$$e_s = H(t_i) - H_1^*(t_i)$$

$$e_s = \begin{pmatrix} 1,000 \\ 1,600 \\ 1,400 \\ 0,600 \\ 0,200 \\ 0,800 \end{pmatrix} - \begin{pmatrix} 0,933 \\ 1,415 \\ 1,415 \\ 0,933 \\ 0,451 \\ 0,451 \end{pmatrix} = \begin{pmatrix} 0,0670 \\ 0,185 \\ 0,154 \\ 0,333 \\ 0,251 \\ 0,349 \end{pmatrix} \rightarrow \|e_s\|_\infty = \max_{1 \le i \le 6} |e_s| = 0,35$$

$$r_\infty = \frac{\|e\|_\infty}{\|H(t)\|_\infty} \cong 21,8\ \%$$

- c) Utilizar ahora la función $H2*(t) = ho + a1 \,.\, \sin\left(\frac{2\,\pi\,t}{12}\right) + a2 \,.\, \cos\left(\frac{2\,\pi\,t}{12}\right)$ y repetir b).

$$H_2^*(t) = \underbrace{\overset{c_0}{\overbrace{ho}}\,1}_{\varphi_0} + \overset{c_1}{\overbrace{a_1}}\ \operatorname{sen}\underbrace{\left(\frac{2\pi}{12}\,t\right)}_{\varphi_1} + \overset{c_2}{\overbrace{a_2}}\ \cos\underbrace{\left(\frac{2\pi}{12}\,t\right)}_{\varphi_2}$$

$n = 2$

$m = 5$

$$\varphi_0 = (1;1;1;1;1;1)$$

$$\varphi_1 = \left(0;\frac{\sqrt{3}}{2};\frac{\sqrt{3}}{2};0;-\frac{\sqrt{3}}{2};-\frac{\sqrt{3}}{2}\right)$$

$$\varphi_2 = \left(1;\frac{1}{2};-\frac{1}{2};-1;-\frac{1}{2};\frac{1}{2}\right)$$

$$f = (1{,}0;1{,}6;1{,}4;0{,}6;0{,}2;0{,}8)$$

$$\begin{pmatrix} \langle \varphi_0\,\varphi_0 \rangle & \langle \varphi_0\,\varphi_1 \rangle & \langle \varphi_0\,\varphi_2 \rangle \\ \langle \varphi_1\,\varphi_0 \rangle & \langle \varphi_1\,\varphi_1 \rangle & \langle \varphi_1\,\varphi_2 \rangle \\ \langle \varphi_2\,\varphi_0 \rangle & \langle \varphi_2\,\varphi_1 \rangle & \langle \varphi_2\,\varphi_2 \rangle \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \langle H\,\varphi_0 \rangle \\ \langle H\,\varphi_1 \rangle \\ \langle H\,\varphi_2 \rangle \end{pmatrix}$$

$$\langle \varphi_0\, \varphi_0 \rangle = 6$$

$$\langle \varphi_0\, \varphi_1 \rangle = \langle \varphi_1\, \varphi_0 \rangle = 0$$

$$\langle \varphi_0\, \varphi_2 \rangle = \langle \varphi_2\, \varphi_0 \rangle = 0$$

$$\langle \varphi_1\, \varphi_1 \rangle = 3$$

$$\langle \varphi_1\, \varphi_2 \rangle = \langle \varphi_2\, \varphi_1 \rangle = 0$$

$$\langle \varphi_2\, \varphi_2 \rangle = 3$$

$$\langle H\, \varphi_0 \rangle = 5{,}6.$$

$$\langle H\, \varphi_1 \rangle = \sqrt{3}$$

$$\langle H\, \varphi_2 \rangle = 0{,}8$$

$$\Rightarrow \begin{pmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 5{,}6 \\ \sqrt{3} \\ 0{,}8 \end{pmatrix} \rightarrow \begin{array}{l} c_0 = h_0 = 0{,}933 \\ c_1 = a_1 = 0{,}577 \\ c_2 = a_2 = 0{,}267 \end{array}$$

$$\boxed{H_2^*(t) = 0{,}933 + 0{,}577 \operatorname{sen}\left(\frac{2\pi}{12} t\right) + 0{,}267 \cos\left(\frac{2\pi}{12} t\right)}$$

$$e_s = H(t_i) - H_2^*(t_i) \longrightarrow \|e_s\|_\infty = \max_{1 \le i \le 6} |e_s| = 0{,}22$$

$$r_\infty = \frac{0{,}216}{1{,}6} \cong 13{,}5\,\%$$

Eje4) Obtener una fórmula del tipo $P(x) = a\,e^{m\,x}$ a partir de los datos que siguen:

x	1	2	3	4
$P(x)$	7	11	17	27

$$P^*(x) = a\,e^{mx} \longrightarrow \ln(P^*(x)) = \overbrace{\ln(a)}^{c_0}\underbrace{1}_{\varphi_0} + \overbrace{m}^{c_1}\underbrace{x}_{\varphi_1}$$

$$f^* = c_0 \cdot 1 + c_1 \cdot x \qquad n = 2 \;;\; m = 3$$

$$\varphi_0 = (1;1;1;1)$$

$$\varphi_1 = (1;2;3;4)$$

$$f = \underline{\ln(P(x))} = (\ln(7);\ln(11);\ln(17);\ln(27))$$

$$<\varphi_0\,\varphi_0> = 4 \qquad <f\,\varphi_0> = 10{,}47$$

$$<\varphi_0\,\varphi_1> = <\varphi_1\,\varphi_0> = 10 \qquad <f\,\varphi_1> = 28{,}42$$

$$<\varphi_1\,\varphi_1> = 30$$

$$\langle \varphi_0\, \varphi_0 \rangle = 4$$
$$\langle \varphi_0\, \varphi_1 \rangle = \langle \varphi_1\, \varphi_0 \rangle = 10$$
$$\langle \varphi_1\, \varphi_1 \rangle = 30$$
$$\langle f\, \varphi_0 \rangle = 10{,}47$$
$$\langle f\, \varphi_1 \rangle = 28{,}42$$

$$\Longrightarrow \begin{pmatrix} \langle \varphi_0\, \varphi_0 \rangle & \langle \varphi_0\, \varphi_1 \rangle \\ \langle \varphi_1\, \varphi_0 \rangle & \langle \varphi_1\, \varphi_1 \rangle \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \langle f\, \varphi_0 \rangle \\ \langle f\, \varphi_1 \rangle \end{pmatrix}$$

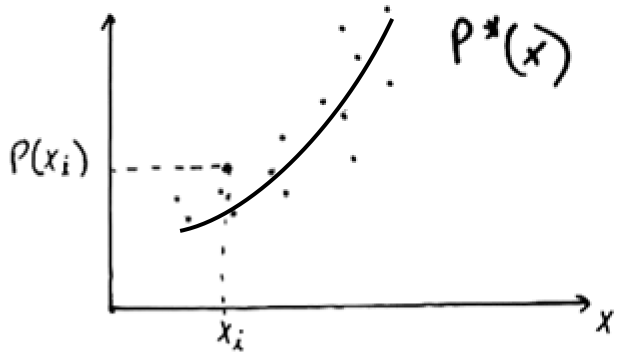
$$\begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 10{,}47 \\ 28{,}42 \end{pmatrix}$$

$$c_0 = \underline{\ln(a)} = 1{,}495 \rightarrow a = 4{,}459$$

$$c_1 = m = 0{,}4490$$

$$\Rightarrow \boxed{P^*(x) = 4{,}459\, e^{0{,}4490x}}$$

$$P^*(x) = c_0 e^{c_1 x}$$



$$e = \sum_{i=0}^{m} e_i^2 = \sum_{i=0}^{m} \left(P_i - P_i^*\right)^2 = \sum_{i=0}^{m} \left(P_i - c_0 e^{c_1 x_i}\right)^2$$

$$\downarrow$$

$$\frac{\partial e}{\partial c_k} = 0 \qquad k = 0;1$$

$$\left.\begin{aligned}
\frac{\partial e}{\partial c_0} &= \sum_{i=0}^{m} 2\left(P_i - c_0 e^{c_1 x_i}\right)\left(-e^{c_1 x_i}\right) = 0 \\
\frac{\partial e}{\partial c_1} &= \sum_{i=0}^{m} 2\left(P_i - c_0 e^{c_1 x_i}\right)\left(-c_0 x_i e^{c_1 x_i}\right) = 0
\end{aligned}\right\} \text{SENL}$$

→ Ajuste de funciones continuas

$$f^*(x) = P_m(x) = \sum_{k=0}^{m} a_k x^k$$

$$e = \int_a^b \left[f(x) - P_m(x) \right]^2 dx$$

$$\downarrow$$

$$\frac{de}{da_j} = 0 \qquad j = 0, \dots, m$$

$$\downarrow$$

$$\sum_{k=0}^{m} \underbrace{a_k \int_a^b x^{k+j} \, dx}_{H_{j+1,k+1}} = \underbrace{\int_a^b f(x) \, x^j \, dx}_{r_{j+1}} \qquad j = 0, \dots, m$$

$$\bar{\bar{H}} \cdot \bar{a} = \bar{r} \longrightarrow H_{j+1,k+1} = \frac{b^{j+k+1} - a^{j+k+1}}{j+k+1} \qquad \begin{matrix} j = 0, \dots, m \\ k = 0, \dots, m \end{matrix}$$

$$r_{j+1} = \int_a^b f(x) \, x^j \, dx$$

Eje 9) Encontrar la aproximación polinómica de grado 2 de f(x) en el intervalo [0,1]

$$f(x) = e^x \qquad [0,1] \qquad \rightarrow \qquad f^*(x) = P_2(x) = a_0 + a_1 x + a_2 x^2 \qquad m=2$$

$$\begin{array}{cc} & \begin{array}{ccc} k=0 & k=1 & k=2 \end{array} \\ \begin{array}{c} j=0 \\ j=1 \\ j=2 \end{array} & \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix} \end{array} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

$$H_{j+1,k+1} = \frac{b^{j+k+1} - a^{j+k+1}}{j+k+1}$$

$$r_{j+1} = \int_a^b f(x)\, x^j \, dx$$

$$H_{11} = \frac{b^1 - a^1}{1} = \frac{1-0}{1} = 1$$

$$H_{12} = \frac{1^2 - 0}{2} = \frac{1}{2} = H_{21}$$

$$H_{13} = \frac{1^3 - 0^3}{3} = \frac{1}{3} = H_{31}$$

$$H_{22} = \frac{1^3 - 0^3}{3} = \frac{1}{3}$$

$$H_{33} = \frac{1^5 - 0^5}{5} = \frac{1}{5}$$

$$H_{23} = \frac{1^4 - 0^4}{4} = \frac{1}{4} = H_{32}$$

$$r_1 = \int_0^1 e^x x^0 \, dx = e^x \Big|_0^1 = 1{,}718$$

$$r_2 = \int_0^1 e^x x^1 \, dx = e^x (x-1) \Big|_0^1 = 1$$

$$r_3 = \int_0^1 e^x x^2 \, dx = e^x (x^2 - 2x + 2) \Big|_0^1 = 0{,}718$$

$$\begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1{,}718 \\ 1 \\ 0{,}7183 \end{pmatrix}$$

$a_0 = 1{,}013$

$a_1 = 0{,}8511$

$a_2 = 0{,}8392$

$$\boxed{f^*(x) = P_2(x) = 1{,}013 + 0{,}8511\, x + 0{,}8392\, x^2}$$