Resolver

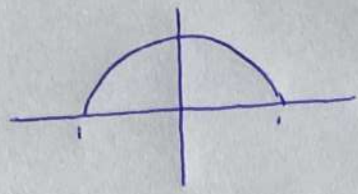
$\nabla^2 u(\rho,\varphi)=0$

$\rho \in (0,1)$

$\varphi \in (0,\pi)$

$$\begin{cases} u(1,\varphi) = \operatorname{sen}(\varphi) \\ u_\varphi(\rho,0)=0 \\ u_\varphi(\rho,\pi)=0 \end{cases}$$

+ Recinto acotado



Laplace en Polares:

$$\nabla^2 u(\rho,\varphi) = u_{rr} + \frac{u_r}{r} + \frac{u_{\varphi\varphi}}{r^2} = 0 \quad (1)$$

Propongo para resolver

$$u(\rho,\varphi) = R(r)\,\varphi(\varphi) \quad (2)$$

Condiciones de Contorno:

$u(1,\varphi) = R(1)\varphi(\varphi) = \operatorname{sen}(\varphi)$

$u_\varphi(\rho,0) = R(r)\varphi'(0) = 0 \Rightarrow \varphi'(0)=0$ I

$u_\varphi(\rho,\pi) = R(r)\varphi'(\pi) = 0 \Rightarrow \varphi'(\pi)=0$ II

$|u(\rho,\varphi)| < M$ La agrego

(2) en (1)

$$R''(r)\varphi(\varphi) + \frac{R'(r)\varphi(\varphi)}{r} + \frac{\varphi''(\varphi)R(r)}{r^2} = 0$$

Dividiendo x $R(r)\varphi(\varphi)$ y operando

$$r^2\frac{R''(r)}{R(r)} + r\frac{R'(r)}{R(r)} = -\frac{\varphi''(\varphi)}{\varphi(\varphi)} = \lambda$$

$$\begin{cases} \varphi''(\varphi) + \lambda\varphi(\varphi) = 0 \quad (3) \\ r^2R''(r) + rR'(r) - \lambda R(r) = 0 \quad (4) \end{cases}$$

Junto I y II $\Rightarrow$ PSLS

$\varphi(\varphi) \to \lambda=0 \quad \varphi''(\varphi)=0 \quad \to \varphi(\varphi) = A_1\varphi + A_0$

$\to \varphi'(\varphi) = A_1 \quad \to \varphi'(\pi) = 0 = A_1$

$\} \ \varphi(\varphi) = A_0$

$\lambda > 0$

$\varphi(\varphi) = B_0\cos(\sqrt{\lambda}\varphi) + B_1\operatorname{sen}(\varphi\sqrt{\lambda}) \qquad \varphi'(0) \Rightarrow B_1 = 0$

$\varphi'(\varphi) = \sqrt{\lambda}\left(-B_0\operatorname{sen}(\sqrt{\lambda}\varphi) + B_1\cos(\varphi\sqrt{\lambda})\right) \to \varphi'(\pi) = 0 = -B_0\operatorname{sen}(\sqrt{\lambda}\varphi) \qquad \sqrt{\lambda}\varphi = n\pi$

$\boxed{n=\sqrt{\lambda}}$

$\varphi_n = B_n\cos(n\varphi)$

Para $R(r)$ — Por recinto acotado

$\lambda=0 \qquad R(r) = C_0\ln(r) + C_1 \qquad r\to 0, \ \ln(r)\to\infty \quad \to C_0 = 0 \quad \to R(r) = C_1$

$\lambda>0 \qquad R(r) = D_0 r^n + D_1 r^{-n} \qquad r\to 0, \ r^{-n}\to\infty \quad \to D_1 = 0$

$R(r) = D_n r^n$

Luego $u(r,\varphi) = \underbrace{C_1 * A_0}_{F_0} + \sum_{n=1}^{\infty} \underbrace{B_n\cos(n\varphi) D_n}_{F_n} r^n$

$u(1,\varphi) = \operatorname{sen}(\varphi) = F_0 + \sum_{n=1}^{\infty} F_n\cos(n\varphi) \longrightarrow$

obs $\operatorname{sen}(\varphi) \to$ impar

$a_0 = a_n = 0$

La debo desarrollar como par

$$f(x) = \begin{cases} \operatorname{sen}(x) & \varphi\in(0,2\pi) \\ -\operatorname{sen}(x) & \varphi\in(-2\pi,0) \end{cases}$$

$$F_0 = \frac{2}{T}\int_0^{2\pi}\operatorname{sen}(\varphi)\,d\varphi = \frac{2*2}{2\pi}\int_0^{\pi}\operatorname{sen}(\varphi)\,d\varphi = \frac{4}{\pi}$$

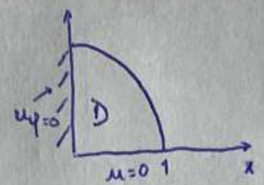
$$F_n = \frac{2}{2\pi} * 2\int_0^{\pi}\operatorname{sen}(\varphi)\cos(n\pi)\,d\varphi = -\frac{2}{\pi}\left[\frac{\cos[\pi(1+n)]-1}{1+n} + \frac{\cos[\pi(1-n)]-1}{1-n}\right]$$

Resolver: $\nabla^2 u(\rho,\varphi)=0$

$0<\rho<1$

$0<\varphi<\pi/2$

$$\begin{cases} u(\rho,0)=0 \\ u'_\varphi(\rho,\pi/2)=0 \\ u(1,\varphi)=\operatorname{sen}(2\varphi) \end{cases}$$



Ec de Laplace

$$\nabla^2 u(\rho,\varphi) = u_{rr} + \frac{u_r}{r} + \frac{u_{\varphi\varphi}}{r^2}$$

Condiciones de Contorno

$u(\rho,0) = R(\rho)\varphi(0)=0 \rightarrow \varphi(0)=0$

$u'_\varphi(\rho,\pi/2) = R(\rho)\varphi'(\frac{\pi}{2}) \rightarrow \varphi'(\frac{\pi}{2})=0$

$u(1,\varphi) = R(1)\varphi(\varphi) = \operatorname{sen}(2\varphi)$

¿La otra condición es que el mismo esta acotado?

Propongo separación de variable

$u(\rho,\varphi) = R(\rho)\varphi(\varphi)$

$u_\rho = R'(\rho)\varphi(\varphi) \quad \wedge \quad u_\varphi = R(\rho)\varphi'(\varphi)$

$u_{rr} = R''(\rho)\varphi(\varphi) \quad \wedge \quad u_{\varphi\varphi} = R(\rho)\varphi''(\varphi)$

Reemplazando en ec de Laplace

$$\nabla^2 u(\rho,\varphi) = R''(\rho)\varphi(\varphi) + \frac{R'(\rho)\varphi(\varphi)}{r} + \frac{\varphi''(\varphi)R(r)}{r^2} = 0$$

dividiendo por $R(r)\varphi(\varphi)$

$$\Rightarrow \frac{R''(\rho)}{R(r)\cancel{\varphi(\varphi)}} + \frac{R'(\rho)}{rR(r)} + \frac{\varphi''(\varphi)}{\varphi(\varphi)r^2} = 0$$

$$\Rightarrow \frac{r^2R''(r)}{R(r)} + \frac{rR'(r)}{R(r)} = -\frac{\varphi''(\varphi)}{\varphi(\varphi)} = \lambda$$

$$\begin{cases} \varphi''(\varphi) + \lambda\varphi(\varphi) = 0 & (1) \\ r^2R''(r) + rR'(r) - \lambda R(r) = 0 & (2) \end{cases}$$

Resolviendo $\varphi(\varphi)$

Para $\lambda=0$

$\varphi''(\varphi)=0 \rightarrow \varphi(\varphi) = A_0\varphi + A_1 \rightarrow \varphi(0)=0=A_1 \rightarrow \cup$ $\quad\lambda=0 \notin$ autovalor

$\rightarrow \varphi'(\pi/2) = A_0 = 0 \rightarrow A_0=0$

Para $\lambda>0$

$\rightarrow \varphi(\varphi) = B_0\cos(\sqrt{\lambda}\varphi) + B_1\operatorname{sen}(\sqrt{\lambda}\varphi)$

$\rightarrow \varphi(0) = B_0 = 0$

$\rightarrow \varphi'(\pi/2) = +B_1\cos(\sqrt{\lambda}\,\pi/2) = 0 \qquad \sqrt{\lambda}\frac{\pi}{2} = (2n+1)\frac{\pi}{2} \quad \rightarrow \sqrt{\lambda} = 2n+1$

$\rightarrow \varphi(\varphi) = B_1\operatorname{sen}((2n+1)\varphi)$

¿No debería llegar a $\lambda=0 \notin$ autovalor?

Resolviendo $R(r)$

$\lambda=0 \qquad R(r) = C_0\ln|r| + D_0 \qquad R$ Acotado $\quad r\to 0, \ \ln\to\infty \quad \rightarrow C_0=0 \rightarrow R_0(r)=D_0$

¿?

$\lambda>0 \qquad R(r) = C_0 r^{\sqrt{-\lambda}} + D_0 r^{-\sqrt{-\lambda}} \qquad -\lambda = \lambda$

$\sqrt{\lambda} = (2n+1)$

$R(r) = C_0 r^{(2n+1)} + D_0 r^{-(2n+1)}$

$r\to 0 \quad r^{-(2n+1)} \to \infty \quad \notin$ Acotado $\Rightarrow D_0=0$

$R(r) = C_0 r^{(2n+1)}$

Entonces

$$u(\rho,\varphi) = \sum_{n=0}^{\infty} B_1 \, sen((2n+1)\varphi) \cdot C_0 \, r^{(2n+1)}$$

$$u(1,\varphi) = \sum_{n=0}^{\infty} D_n \, sen((2n+1)\varphi) = sen(2\varphi)$$

Desarrollo serie de Fourier para hallar D_n

$$D_n = \frac{2}{T}\int_{\alpha}^{\alpha+T} f(x)\, sen\left(\frac{2\pi}{T} n x\right) dx \Rightarrow \cancel{T} \cancel{\Rightarrow} \quad \frac{2\pi n}{T} = 2n \Rightarrow T = \pi$$

$$D_n = \frac{4}{\pi}\int_0^{\pi/2} \underbrace{\underset{impar}{sen(2\varphi)} \cdot \underset{impar}{sen[(2n+1)\varphi]}}_{par} \, d\varphi$$

Por tabla

$$\int sen \cdot sen = \frac{4}{\pi}\left\{ \frac{1}{2}\left[\frac{sen[(2-(2n+1))\varphi]}{2-(2n+1)} - \frac{sen[(2+2n+1)\varphi]}{2+2n+1} \right] \right\} \Bigg|_0^{\pi/2}$$

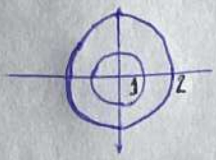
$$= \frac{2}{\pi}\left[\frac{(-1)^n}{1-2n} - \frac{(-1)^n}{3+2n} \right]$$

Finalmente

$$u(\rho,\varphi) = \sum_{n=0}^{\infty} \frac{2}{\pi}\left[\frac{(-1)^n}{1-2n} - \frac{(-1)^n}{3+2n} \right] \cdot sen[(2n+1)\varphi] \, r^{(2n+1)}$$

Ec de Laplace en Polares

$$\begin{cases} \nabla^2 u = 0 \\ u(1,\varphi) = 0 \\ u(2,\varphi) = sen(\varphi/2) \end{cases}$$



Propongo sep. variables . Laplace en polares $\rightarrow \nabla^2 u(r,\varphi) = 0 = u_{rr} + \frac{u_r}{r} + \frac{u_{\varphi\varphi}}{r^2}$

$u(r,\varphi) = R(r)\varphi(\varphi)$

$u(1,\varphi) = R(1)\varphi(\varphi) = 0 \Rightarrow R(1) = 0$

Por características del recinto

$u(r,0) = u(r,2\pi) \Rightarrow R(r)\varphi(0) = R(r)\varphi(2\pi) \Rightarrow \varphi(0) = \varphi(2\pi)$ I

$\Rightarrow$ Lo mismo $u_\varphi \rightarrow \varphi'(0) = \varphi'(2\pi)$ II

Luego, reemplazando en Laplace

$$R''(r)\varphi(\varphi) + \frac{R'(r)\varphi(\varphi)}{r} + \frac{\varphi''(\varphi)R(r)}{r^2} = 0 \Rightarrow \frac{R''(r)}{R(r)} + \frac{R'(r)}{R(r)r} + \frac{\varphi''(\varphi)}{\varphi(\varphi)r^2} = 0$$

$$\Rightarrow r^2\frac{R''(r)}{R(r)} + \frac{R'(r)\,r}{R(r)} = -\frac{\varphi''(\varphi)}{\varphi(\varphi)} = +\lambda$$

$$\begin{cases} \varphi''(\varphi) + \overset{\lambda}{\wedge}\varphi(\varphi) = 0 \quad (1) \\ r^2 R''(r) + rR'(r) - \lambda R(r) = 0 \end{cases}$$

1 junto II $\wedge$ I $\Rightarrow$ PSLS

$\lambda = 0$ $\quad \varphi(\varphi) = A_0\varphi + A_1 \rightarrow \varphi(0) = A_1$, $\varphi(2\pi) = A_0 * 2\pi + A_1 \Rightarrow A_1 = A_0 * 2\pi + A_1$, $A_0 = 0$

$\varphi'(\varphi) = A_0$

$\varphi(\varphi) = A_1$

Luego $\lambda > 0$

$\varphi(\varphi) = B_0 \cos(\sqrt{\lambda}\,\varphi) + B_1 sen(\varphi\sqrt{\lambda}) \rightarrow$ $\varphi(0) = B_0$, $\varphi(2\pi) = B_0\cos(\sqrt{\lambda}2\pi) + B_1 sen(\sqrt{\lambda}2\pi)$

$\varphi'(\varphi) = \sqrt{\lambda}\,[-sen(\sqrt{\lambda}\varphi) * B_0 + B_1 \cos(\sqrt{\lambda}\varphi) \rightarrow$ $\varphi'(0) = \sqrt{\lambda}B_1$

$\varphi'(2\pi) = \sqrt{\lambda}\,[-sen(\sqrt{\lambda}2\pi) * B_0 + B_1\cos(\sqrt{\lambda}2\pi)$

$B_0 = B_0\cos(\sqrt{\lambda}2\pi) + B_1 sen(\sqrt{\lambda}2\pi)$

$\cancel{\sqrt{\lambda}}B_1 = \cancel{\sqrt{\lambda}}\,[-sen(\sqrt{\lambda}2\pi) * B_0 + B_1\cos(\sqrt{\lambda}2\pi)$

det $\neq 0$

$$\begin{Bmatrix} 1 - \cos(\sqrt{\lambda}2\pi) & -sen(\sqrt{\lambda}2\pi) \\ sen(\sqrt{\lambda}2\pi) & 1 - \cos(\sqrt{\lambda}2\pi) \end{Bmatrix} \begin{pmatrix} B_0 \\ B_1 \end{pmatrix}$$

$$(1 - \cos(\sqrt{\lambda}2\pi))^2 + sen(\sqrt{\lambda}2\pi)^2 = 0$$

$$1 - 2\cos(\sqrt{\lambda}2\pi) + 1 = 0$$

$(1 - \cos(a))(1 - \cos(a))$

$1 - 2\cos(a) + \cos(a)^2$

$$\cos(\sqrt{\lambda}2\pi) = 1 \Rightarrow \sqrt{\lambda}2\pi = 2n\pi$$

$$n = \sqrt{\lambda}$$

Luego $\varphi_n(\varphi) = A_n \cos(n\pi) + B_n sen(n\pi)$

Resolver: $\nabla^2 u(\rho,\varphi)=0$ $\quad u(\rho,0)=0$

Para $R(r)$

$\lambda=0 \quad R_0(r)=C_0\ln(r)+D_0$ $\qquad \lambda>0 \quad R(r)=C_1 r^{\sqrt{\lambda}}+D_1 r^{-\sqrt{\lambda}}$

$R_0(1)=D_0=0$ $\qquad R(1)=C_1+D_1=0 \rightarrow C_1=-D_1$

$R_0(r)=C_0\ln(r)$

$R(r)=C_1 r^n - C_1 r^{-n} = C_n\{r^n-r^{-n}\}$

Entonces

$$u(r,\varphi)=A_1 * C_0\ln(r) + \sum_{n=1}^{\infty}\left(A_n\cos(n\varphi)+B_n\,\mathrm{sen}(n\varphi)\right) * C_n\{r^n-r^{-n}\}$$

$$u(2,\varphi)=\underbrace{A_1C_0\ln(2)}_{F_0}+\sum_{n=1}^{\infty}\{A_n\cos(n\varphi)+B_n\,\mathrm{sen}(n\varphi)\}\cdot C_n\{2^n-2^{-n}\}=\mathrm{sen}\left(\frac{\varphi}{2}\right)$$

$$F_0+\sum F_n\cos(n\pi)+G_n\,\mathrm{sen}(n\varphi)=\mathrm{sen}\left(\frac{\varphi}{2}\right)$$

$\frac{2\pi}{T}m\varphi = n\varphi$

$T=\pi 2$

$$F_0=\frac{1}{2\pi}*2\int_0^{\pi}\mathrm{sen}\left(\frac{\varphi}{2}\right)d\varphi = \frac{1}{\pi}*\left(-2\cos\left(\frac{\varphi}{2}\right)\right)\Big|_0^{\pi} = \frac{1}{\pi}\left\{-(-2)\right\}=\boxed{\frac{2}{\pi}}$$

$\frac{4}{2\pi}\ 2$

Luego $A_1C_0=\dfrac{2}{\pi\ln(2)}$

$$G_n=\frac{2}{2\pi}*2\int_0^{\pi}\underset{\text{Impar}}{\mathrm{sen}(n\varphi)}\ \underset{\text{Impar}}{\mathrm{sen}\left(\frac{\varphi}{2}\right)}d\varphi=\frac{\cancel{2}}{\pi}\left[\frac{\mathrm{sen}\left(\left(\frac{1}{2}-n\right)\varphi\right)}{\cancel{2}\left(\frac{1}{2}-n\right)}-\frac{\mathrm{sen}(1/2+n)}{\cancel{2}\left(\frac{1}{2}+n\right)}\right]$$

$$=\pi\left\{\frac{\mathrm{sen}\left\{\left(\frac{1}{2}-n\right)\pi\right\}}{\left(\frac{1}{2}-n\right)}-\frac{\mathrm{sen}(1/2+n)}{\left(\frac{1}{2}+n\right)}\right\}$$

Fue, Pinche acá

$$F_0=\frac{\int_{-\pi}^{\pi}\mathrm{sen}\left(\frac{\varphi}{2}\right)*1\,dx}{\int_{-\pi}^{\pi}1\,dx}=\frac{1}{2\pi}\quad \frac{42}{2\pi}$$