① **Resolver** $\displaystyle\int_0^{+\infty} f(t)\,\mathrm{sen}(wt)\,dt = \underbrace{\frac{1-\cos w}{w}}_{\text{Transformada seno.}}$

Recordemos la definición de Transformada de Fourier,

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt.$$

Separando,

$$F(w) = \int_{-\infty}^{\infty} f(t)\cos(wt)\,dt - i\int_{-\infty}^{\infty} f(t)\,\mathrm{sen}(wt)\,dt$$

observemos que si $f(t)$ ∈ impar:

$$F(w) = -2i\int_{0}^{\infty} f(t)\,\mathrm{sen}(wt)\,dt$$

Entonces, la Transformada de Fourier de $f(t)$ será igual a 2 veces la TFS multiplicado por "$-2i$".

Dado que $f(t)$ ∈ impar la antitransformada la puedo escribir como,

$$f^*(t) = \frac{i}{\pi}\int_0^{\infty} F(w)\,\mathrm{sen}(wt)\,dt \qquad (1)$$

o bien puedo no simplificar y adopto el camino,

$$f^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(w)\,e^{iwt}\,dt \qquad (2)$$

uso (2),

$$f^*(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-2i)\left(\frac{1-\cos w}{w}\right) e^{iwt}\,dt.$$

$$f^*(t) = -\frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{e^{iwt}}{w} - \frac{\cos w}{w}\,e^{iwt}\right) dt$$

$$f^*(t) = -\frac{i}{\pi}\left[\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega}\,d\omega - \int_{-\infty}^{\infty} \frac{\cos\omega}{\omega}\, e^{i\omega t}\,d\omega\right.$$

$$f^*(t) = -\frac{i}{\pi}\left[\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega}\,d\omega - \int_{-\infty}^{\infty} \frac{e^{i\omega}+e^{-i\omega}}{2\omega}\, e^{i\omega t}\,d\omega\right]$$

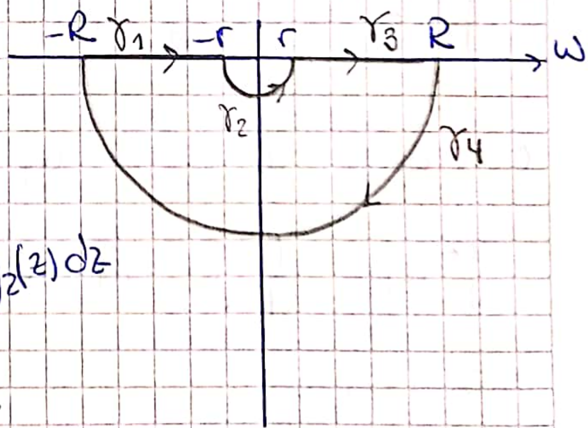
$$f^*(t) = -\frac{i}{\pi}\left[\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega}\,d\omega - \int_{\infty}^{\infty} \frac{e^{i\omega(t+1)}}{2\omega}\,d\omega + \int_{\infty}^{\infty} \frac{e^{i\omega(t-1)}}{2\omega}\,d\omega\right]$$

$$f^*(t) = \underbrace{-\frac{i}{\pi}\left[\int_{-\infty}^{\infty} \frac{e^{i\omega t}}{\omega}\,d\omega\right]}_{1} + \underbrace{\frac{i}{2\pi}\int_{\infty}^{\infty} \frac{e^{i\omega(t+1)}}{\omega}\,d\omega}_{2} - \underbrace{\frac{i}{2\pi}\int_{\infty}^{\infty} \frac{e^{i\omega(t-1)}}{\omega}\,d\omega}_{3}$$

$$g_1(z) = \frac{e^{izt}}{z} \quad ; \quad g_2(z) = \frac{e^{iz(t+1)}}{z} \quad ; \quad g_3(z) = \frac{e^{iz(t-1)}}{z}$$

<u>Caso $t < -1$</u>

camino:



- $g_2(z) = \dfrac{e^{iz(t+1)}}{z}$

$\Gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$

$$\oint_\Gamma = \int_{\gamma_1} g_2(z)\,dz + \int_{\gamma_2} g_2(z)\,dz + \int_{\gamma_3} g_2(z)\,dz$$

$$+ \int_{\gamma_4} g_2(z)\,dz = 2\pi i \,\mathrm{Re} = 0 .$$

En los caminos 1 y 3 $z = \omega$ y entonces la suma de las dos integrales con $R \to \infty$ y $r \to 0$ me da la integral que quiero calculo

$$\lim_{\substack{r \to 0 \\ R \to \infty}} \int_{\gamma_1} g_2(z)\,dz + \int_{\gamma_3} g_2(z)\,dz = \int_{-\infty}^{\infty} g_2(\omega)\,d\omega$$

Para la integral en γ_2:

$z=0 \in$ Polo de 1^{er} orden
$\gamma_r: z_0 + re^{i\varphi}$

$$\Rightarrow \lim_{r\to 0} \int_{\gamma_r} f(z)dz = i\alpha \operatorname{Res}[f(z); z_0]$$

$$\int_{\gamma_2} g_2(z)dz = \pi . i . \frac{e^{iz(t+1)}}{1}\Big|_{z=0} = \pi i$$

Por el Teorema de Jordan $\int_{\gamma_4} = 0$.

$$\int_{-\infty}^{\infty} g_2(w)dw = -\int_{\gamma_2} g_2(z)dz = -i\pi$$

- $g_3(z) = \dfrac{e^{iz(t-1)}}{z}$

El calculo para $g_3(z)$ es similar a $g_2(z)$ con la dif de que ~~pero dado~~ que ahora $m = (t-1)$. Pero dado que estamos analizando el caso en el que $t<-1$ ~~too~~ también se implicará que $(t-1)<0$.

Por lo tanto, el resultado ~~es~~ de la integral para $g_3(z)$ será el mismo que para ~~g_2~~ $g_2(z)$.

$$\int_{-\infty}^{\infty} g_3(w)\,dw = -i\pi.$$

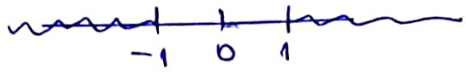
- $g_1(z) = \dfrac{e^{izt}}{z}$

Tratamiento similar:

$$\int_{-\infty}^{\infty} g_1(w)dw = -\int_{\gamma_2} g_1(z)dz = -\pi i\left[e^{izt}\Big|_{z=0}\right] = -\pi i$$

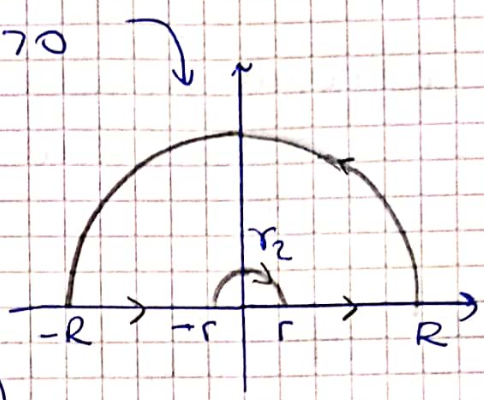
→ Junto:

$$f^*(t) = -\frac{i}{\cancel{\pi}}(-i)\cancel{\pi} + \cancel{\frac{i}{2\pi}.(-i\pi)} - \cancel{\left[\frac{i}{2\pi}(-i\pi)\right]} = -1 \text{ con } t<-1$$



caso ~~$t>-1$~~ $-1<t<1$

- $g_2(z) = \dfrac{e^{iz(t+1)}}{z}$ $\Rightarrow (t+1)>0$



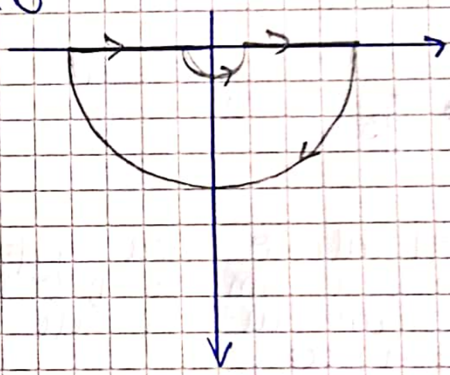
Similar planteo $\therefore$

$$\int_{-\infty}^{\infty} g_2(w)\,dw = -\int_{\gamma_2} g_2(z)\,dz$$

$$= -\left(-i\pi\, e^{iz(t+1)}\Big|_{z=0}\right)$$

$$= i\pi .$$

- $g_3(z) = \dfrac{e^{iz(t-1)}}{z}$ $\Rightarrow (t-1)<0$



$\therefore$ $$\int_{-\infty}^{\infty} g_3(w)\,dw = -i\pi$$

- $g_1(z) = \dfrac{e^{izt}}{z}$ $\rightsquigarrow$ acá debería dividir el intervalo:

$-1<t<0$ (A) $\quad$ $0<t<1$ (B)

No sé cómo continuar

caso $t > 1$

- $g_2(z) = \dfrac{e^{iz(T+1)}}{z} \Rightarrow \cancel{t>1}\; t+1 > 0$

$$\int_{-\infty}^{\infty} g_2(w)\,dw = i\pi$$

- $g_3(z) = \dfrac{e^{iz(t-1)}}{z} \Rightarrow t-1 > 0$

$$\int_{-\infty}^{\infty} g_3(w)\,dw = i\pi.$$

- $g_1(z) = \dfrac{e^{izt}}{z} \Rightarrow t > 0$

$$\int_{-\infty}^{\infty} g_1(w)\,dw = \cancel{i\pi}$$

¡Debería compensarse! y darme 0 para $t > 1$.