1

Dynamics of First-Order Difference Equations

1.1 Introduction

Difference equations usually describe the evolution of certain phenomena over the course of time. For example, if a certain population has discrete generations, the size of the $(n+1)$st generation $x(n+1)$ is a function of the nth generation $x(n)$. This relation expresses itself in the *difference equation*

$$x(n+1) = f(x(n)). \tag{1.1.1}$$

We may look at this problem from another point of view. Starting from a point x_0, one may generate the sequence

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots.$$

For convenience we adopt the notation

$$f^2(x_0) = f(f(x_0)), \quad f^3(x_0) = f(f(f(x_0))), \quad \text{etc.}$$

$f(x_0)$ is called the *first iterate* of x_0 under f; $f^2(x_0)$ is called the second iterate of x_0 under f; more generally, $f^n(x_0)$ is the nth iterate of x_0 under f. The set of all (positive) iterates $\{f^n(x_0) : n \geq 0\}$ where $f^0(x_0) = x_0$ by definition, is called the (*positive*) *orbit* of x_0 and will be denoted by $O(x_0)$. This iterative procedure is an example of a *discrete dynamical system.* Letting $x(n) = f^n(x_0)$, we have

$$x(n+1) = f^{n+1}(x_0) = f[f^n(x_0)] = f(x(n)),$$

and hence we recapture (1.1.1). Observe that $x(0) = f^0(x_0) = x_0$. For example, let $f(x) = x^2$ and $x_0 = 0.6$. To find the sequence of iterates

$\{f^n(x_0)\}$, we key 0.6 into a calculator and then repeatedly depress the x^2 button. We obtain the numbers

$$0.6, 0.36, 0.1296, 0.01679616, \ldots.$$

A few more key strokes on the calculator will be enough to convince the reader that the iterates $f^n(0.6)$ tend to 0. The reader is invited to verify that for all $x_0 \in (0,1)$, $f^n(x_0)$ tends to 0 as n tends to ∞, and that $f^n(x_0)$ tends to ∞ if $x_0 \notin [-1,1]$. Obviously, $f^n(0) = 0, f^n(1) = 1$ for all positive integers n, and $f^n(-1) = 1$ for $n = 1, 2, 3, \ldots$.

After this discussion one may conclude correctly that difference equations and discrete dynamical systems represent two sides of the same coin. For instance, when mathematicians talk about difference equations, they usually refer to the analytic theory of the subject, and when they talk about discrete dynamical systems, they generally refer to its geometrical and topological aspects.

If the function f in (1.1.1) is replaced by a function g of two variables, that is, $g : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}$, where $\mathbb{Z}^+$ is the set of nonnegative integers and $\mathbb{R}$ is the set of real numbers, then we have

$$x(n+1) = g(n, x(n)). \tag{1.1.2}$$

Equation (1.1.2) is called *nonautonomous* or time-variant, whereas (1.1.1) is called *autonomous* or time-invariant. The study of (1.1.2) is much more complicated and does not lend itself to the discrete dynamical system theory of first-order equations. If an initial condition $x(n_0) = x_0$ is given, then for $n \geq n_0$ there is a *unique solution* $x(n) \equiv x(n, n_0, x_0)$ of (1.1.2) such that $x(n_0, n_0, x_0) = x_0$. This may be shown easily by iteration. Now,

$$\begin{aligned}
&x(n_0+1, n_0, x_0) = g(n_0, x(n_0)) = g(n_0, x_0),\\
&x(n_0+2, n_0, x_0) = g(n_0+1, x(n_0+1)) = g(n_0+1, g(n_0, x_0)),\\
&x(n_0+3, n_0, x_0) = g(n_0+2, x(n_0+2)) = g[n_0+2, g(n_0+1, g(n_0, x_0))].
\end{aligned}$$

And, inductively, we get $x(n, n_0, x_0) = g[n-1, x(n-1, n_0, x_0)]$.

1.2 Linear First-Order Difference Equations

In this section we study the simplest special cases of (1.1.1) and (1.1.2), namely, linear equations. A typical linear *homogeneous* first-order equation is given by

$$x(n+1) = a(n)x(n), \qquad x(n_0) = x_0, \qquad n \geq n_0 \geq 0, \tag{1.2.1}$$

and the associated *nonhomogeneous* equation is given by

$$y(n+1) = a(n)y(n) + g(n), \qquad y(n_0) = y_0, \qquad n \geq n_0 \geq 0, \tag{1.2.2}$$

where in both equations it is assumed that $a(n) \neq 0$, and $a(n)$ and $g(n)$ are real-valued functions defined for $n \geq n_0 \geq 0$.

One may obtain the solution of (1.2.1) by a simple iteration:

$$
\begin{aligned}
x(n_0+1) &= a(n_0)x(n_0) = a(n_0)x_0, \\
x(n_0+2) &= a(n_0+1)x(n_0+1) = a(n_0+1)a(n_0)x_0, \\
x(n_0+3) &= a(n_0+2)x(n_0+2) = a(n_0+2)a(n_0+1)a(n_0)x_0.
\end{aligned}
$$

And, inductively, it is easy to see that

$$
\begin{aligned}
x(n) &= x(n_0+n-n_0)) \\
&= a(n-1)a(n-2)\cdots a(n_0)x_0,
\end{aligned}
$$

$$
\boxed{x(n) = \left[\prod_{i=n_0}^{n-1} a(i)\right] x_0.} \tag{1.2.3}
$$

The unique solution of the *nonhomogeneous* (1.2.2) may be found as follows:

$$
\begin{aligned}
y(n_0+1) &= a(n_0)y_0 + g(n_0), \\
y(n_0+2) &= a(n_0+1)y(n_0+1) + g(n_0+1) \\
&= a(n_0+1)a(n_0)y_0 + a(n_0+1)g(n_0) + g(n_0+1).
\end{aligned}
$$

Now we use mathematical induction to show that, for all $n \in \mathbb{Z}^+$,

$$
\boxed{y(n) = \left[\prod_{i=n_0}^{n-1} a(i)\right] y_0 + \sum_{r=n_0}^{n-1} \left[\prod_{i=r+1}^{n-1} a(i)\right] g(r).} \tag{1.2.4}
$$

To establish this, assume that formula (1.2.4) holds for $n = k$. Then from (1.2.2), $y(k+1) = a(k)y(k) + g(k)$, which by formula (1.2.4) yields

$$
\begin{aligned}
y(k+1) &= a(k)\left[\prod_{i=n_0}^{k-1} a(i)\right] y_0 + \sum_{r=n_0}^{k-1} \left[a(k)\prod_{i=r+1}^{k-1} a(i)\right] g(r) + g(k) \\
&= \left[\prod_{i=n_0}^{k} a(i)\right] y_0 + \sum_{r=n_0}^{k-1} \left(\prod_{i=r+1}^{k} a(i)\right) g(r) \\
&\quad + \left(\prod_{i=k+1}^{k} a(i)\right) g(k) \text{ (see footnote 1)} \\
&= \left[\prod_{i=n_0}^{k} a(i)\right] y_0 + \sum_{r=n_0}^{k} \left(\prod_{i=r+1} a(i)\right) g(r).
\end{aligned}
$$

Hence formula (1.2.4) holds for all $n \in \mathbb{Z}^+$.

[1]Notice that we have adopted the notation $\prod_{i=k+1}^{k} a(i) = 1$ and $\sum_{i=k+1}^{k} a(i) = 0$.

1.2.1 Important Special Cases

There are two special cases of (1.2.2) that are important in many applications. The first equation is given by

$$y(n+1) = ay(n) + g(n), \quad y(0) = y_0. \tag{1.2.5}$$

Using formula (1.2.4) one may establish that

$$\boxed{y(n) = a^n y_0 + \sum_{k=0}^{n-1} a^{n-k-1} g(k).} \tag{1.2.6}$$

The second equation is given by

$$y(n+1) = ay(n) + b, \qquad y(0) = y_0. \tag{1.2.7}$$

Using formula (1.2.6) we obtain

$$y(n) = \begin{cases} a^n y_0 + b\left[\dfrac{a^n - 1}{a - 1}\right] & \text{if } a \neq 1, \\ y_0 + bn & \text{if } a = 1. \end{cases} \tag{1.2.8}$$

Notice that the solution of the differential equation

$$\frac{dx}{dt} = ax(t), \quad x(0) = x_0,$$

is given by

$$x(t) = e^{at} x_0,$$

and the solution of the nonhomogeneous differential equation

$$\frac{dy}{dt} = ay(t) + g(t), \quad y(0) = y_0,$$

is given by

$$y(t) = e^{at} y_0 + \int_0^t e^{a(t-s)} g(s)\, ds.$$

Thus the exponential e^{at} in differential equations corresponds to the exponential a^n and the integral $\int_0^t e^{a(t-s)} g(s)\, ds$ corresponds to the summation $\sum\limits_{k=0}^{n-1} a^{n-k-1} g(k)$.

We now give some examples to practice the above formulas.

Example 1.1. Solve the equation

$$y(n+1) = (n+1)y(n) + 2^n (n+1)!, \qquad y(0) = 1, \qquad n > 0.$$

TABLE 1.1. Definite sum.

Number	Summation	Definite sum
1	$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
2	$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
3	$\sum_{k=1}^{n} k^3$	$\left[\frac{n(n+1)}{2}\right]^2$
4	$\sum_{k=1}^{n} k^4$	$\frac{n(6n^4+15n^3+10n^2-1)}{30}$
5	$\sum_{k=0}^{n-1} a^k$	$\begin{cases} (a^n-1)/(a-1) & \text{if } a \neq 1 \\ n & \text{if } a = 1 \end{cases}$
6	$\sum_{k=1}^{n-1} a^k$	$\begin{cases} (a^n-a)/(a-1) & \text{if } a \neq 1 \\ n-1 & \text{if } a = 1 \end{cases}$
7	$\sum_{k=1}^{n} ka^k,\ a \neq 1$	$\frac{(a-1)(n+1)a^{n+1}-a^{n+2}+a}{(a-1)^2}$

Solution

$$\begin{aligned} y(n) &= \prod_{i=0}^{n-1}(i+1) + \sum_{k=0}^{n-1}\left[\prod_{i=k+1}^{n-1}(i+1)\right]2^k(k+1)! \\ &= n! + \sum_{k=0}^{n-1} n!\,2^k \\ &= 2^n n! \text{ (from Table 1.1)}. \end{aligned}$$

Example 1.2. Find a solution for the equation

$$x(n+1) = 2x(n) + 3^n, \qquad x(1) = 0.5.$$

Solution From (1.2.6), we have

$$\begin{aligned} x(n) &= \left(\frac{1}{2}\right)2^{n-1} + \sum_{k=1}^{n-1} 2^{n-k-1}\,3^k \\ &= 2^{n-2} + 2^{n-1}\sum_{k=1}^{n-1}\left(\frac{3}{2}\right)^k \\ &= 2^{n-2} + 2^{n-1}\frac{3}{2}\left(\frac{\left(\frac{3}{2}\right)^{n-1}-1}{\frac{3}{2}-1}\right) \\ &= 3^n - 5\cdot 2^{n-2}. \end{aligned}$$

Example 1.3. A drug is administered once every four hours. Let $D(n)$ be the amount of the drug in the blood system at the nth interval. The body eliminates a certain fraction p of the drug during each time interval. If the amount administered is D_0, find $D(n)$ and $\lim_{n\to\infty} D(n)$.

Solution We first must create an equation to solve. Since the amount of drug in the patient's system at time $(n+1)$ is equal to the amount at time n minus the fraction p that has been eliminated from the body, plus the new dosage D_0, we arrive at the following equation:

$$D(n+1) = (1-p)D(n) + D_0.$$

Using (1.2.8), we solve the above equation, arriving at

$$D(n) = \left[D_0 - \frac{D_0}{p}\right](1-p)^n + \frac{D_0}{p}.$$

Hence,

$$\lim_{n\to\infty} D(n) = \frac{D_0}{p}. \tag{1.2.9}$$

Let $D_0 = 2$ cubic centimeters (cc), $p = 0.25$.

Then our original equation becomes

$$D(n+1) = 0.75D(n) + 2, \qquad D(0) = 2.$$

Table 1.2 gives $D(n)$ for $0 \le n \le 10$.

It follows from (1.2.9) that $\lim_{n\to\infty} D(n) = 8$, where $D^* = 8\,\text{cc}$ is the equilibrium amount of drug in the body. We now enter the realm of finance for our next example.

Example 1.4. Amortization

Amortization is the process by which a loan is repaid by a sequence of periodic payments, each of which is part payment of interest and part payment to reduce the outstanding principal.

Let $p(n)$ represent the outstanding principal after the nth payment $g(n)$. Suppose that interest charges compound at the rate r per payment period.

The formulation of our model here is based on the fact that the outstanding principal $p(n+1)$ after the $(n+1)$st payment is equal to the outstanding principal $p(n)$ after the nth payment plus the interest $rp(n)$ incurred during the $(n+1)$st period minus the nth payment $g(n)$. Hence

$$p(n+1) = p(n) + rp(n) - g(n),$$

TABLE 1.2. Values of $D(n)$.

n	0	1	2	3	4	5	6	7	8	9	10
$D(n)$	2	3.5	4.62	5.47	6.1	6.58	6.93	7.2	7.4	7.55	7.66

or

$$p(n+1) = (1+r)p(n) - g(n), \qquad p(0) = p_0, \tag{1.2.10}$$

where p_0 is the initial debt. By (1.2.6) we have

$$p(n) = (1+r)^n p_0 - \sum_{k=0}^{n-1} (1+r)^{n-k-1} g(k). \tag{1.2.11}$$

In practice, the payment $g(n)$ is constant and, say, equal to T. In this case,

$$p(n) = (1+r)^n p_0 - ((1+r)^n - 1)\left(\frac{T}{r}\right). \tag{1.2.12}$$

If we want to pay off the loan in exactly n payments, what would be the monthly payment T? Observe first that $p(n) = 0$. Hence from (1.2.12) we have

$$T = p_0 \left[\frac{r}{1-(1+r)^{-n}}\right].$$

Exercises 1.1 and 1.2

1. Find the solution of each difference equation:
   (a) $x(n+1) - (n+1)x(n) = 0, \qquad x(0) = c.$
   (b) $x(n+1) - 3^n x(n) = 0, \qquad x(0) = c.$
   (c) $x(n+1) - e^{2n} x(n) = 0, \qquad x(0) = c.$
   (d) $x(n+1) - \frac{n}{n+1} x(n) = 0, \qquad n \geq 1, \qquad x(1) = c.$
2. Find the general solution of each difference equation:
   (a) $y(n+1) - \frac{1}{2} y(n) = 2, \qquad y(0) = c.$
   (b) $y(n+1) - \frac{n}{n+1} y(n) = 4, \qquad y(1) = c.$
3. Find the general solution of each difference equation:
   (a) $y(n+1) - (n+1)y(n) = 2^n (n+1)!, \qquad y(0) = c.$
   (b) $y(n+1) = y(n) + e^n, \qquad y(0) = c.$
4. (a) Write a difference equation that describes the number of regions created by n lines in the plane if it is required that every pair of lines meet and no more than two lines meet at one point.
   (b) Find the number of these regions by solving the difference equation in case (a).
5. The gamma function is defined as $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, x > 0.$
   (a) Show that $\Gamma(x+1) = x\Gamma(x), \qquad \Gamma(1) = 1.$

(b) If n is a positive integer, show that $\Gamma(n+1) = n!$.

(c) Show that $x^{(n)} = x(x-1)\cdots(x-n+1) = \dfrac{\Gamma(x+1)}{\Gamma(x-n+1)}$.

6. A space (three-dimensional) is divided by n planes, nonparallel, and no four planes having a point in common.

   (a) Write a difference equation that describes the number of regions created.

   (b) Find the number of these regions.

7. Verify (1.2.6).

8. Verify (1.2.8).

9. A debt of \$12,000 is to be amortized by equal payments of \$380 at the end of each month, plus a final partial payment one month after the last \$380 is paid. If interest is at an annual rate of 12% compounded monthly, construct an amortization schedule to show the required payments.

10. Suppose that a loan of \$80,000 is to be amortized by equal monthly payments. If the interest rate is 10% compounded monthly, find the monthly payment required to pay off the loan in 30 years.

11. Suppose the constant sum T is deposited at the end of each fixed period in a bank that pays interest at the rate r per period. Let $A(n)$ be the amount accumulated in the bank after n periods.

    (a) Write a difference equation that describes $A(n)$.

    (b) Solve the difference equation obtained in (a), when $A(0) = 0, T = \$200$, and $r = 0.008$.

12. The temperature of a body is measured as 110° F. It is observed that the amount the temperature changes during each period of two hours is -0.3 times the difference between the previous period's temperature and the room temperature, which is 70° F.

    (a) Write a difference equation that describes the temperature $T(n)$ of the body at the end of n periods.

    (b) Find $T(n)$.

13. Suppose that you can get a 30-year mortgage at 8% interest. How much can you afford to borrow if you can afford to make a monthly payment of \$1,000?

14. Radium decreases at the rate of 0.04% per year. What is its half-life? (The half-life of a radioactive material is defined to be the time needed for half of the material to dissipate.)

15. (Carbon Dating). It has been observed that the proportion of carbon-14 in plants and animals is the same as that in the atmosphere as long as the plant or the animal is alive. When an animal or plant dies, the carbon-14 in its tissue starts decaying at the rate r.
    (a) If the half-life of carbon-14 is 5,700 years, find r.
    (b) If the amount of carbon-14 present in a bone of an animal is 70% of the original amount of carbon-14, how old is the bone?

1.3 Equilibrium Points

The notion of equilibrium points (states) is central in the study of the dynamics of any physical system. In many applications in biology, economics, physics, engineering, etc., it is desirable that all states (solutions) of a given system tend to its equilibrium state (point). This is the subject of study of stability theory, a topic of great importance to scientists and engineers. We now give the formal definition of an equilibrium point.

Definition 1.5. A point x^* in the domain of f is said to be an *equilibrium point* of (1.1.1) if it is a fixed point of f, i.e., $f(x^*) = x^*$.

In other words, x^* is a *constant solution* of (1.1.1), since if $x(0) = x^*$ is an initial point, then $x(1) = f(x^*) = x^*$, and $x(2) = f(x(1)) = f(x^*) = x^*$, and so on.

Graphically, an equilibrium point is the x-coordinate of the point where the graph of f intersects the diagonal line $y = x$ (Figures 1.1 and 1.2). For example, there are three equilibrium points for the equation

$$x(n+1) = x^3(n)$$

where $f(x) = x^3$. To find these equilibrium points, we let $f(x^*) = x^*$, or $x^3 = x$, and solve for x. Hence there are three equilibrium points, $-1, 0, 1$ (Figure 1.1). Figure 1.2 illustrates another example, where $f(x) = x^2 - x + 1$ and the difference equation is given by

$$x(n+1) = x^2(n) - x(n) + 1.$$

Letting $x^2 - x + 1 = x$, we find that 1 is the only equilibrium point.

There is a phenomenon that is unique to difference equations and cannot possibly occur in differential equations. It is possible in difference equations that a solution may not be an equilibrium point but may reach one after finitely many iterations. In other words, a nonequilibrium state may go to an equilibrium state in a finite time. This leads to the following definition.

Definition 1.6. Let x be a point in the domain of f. If there exists a positive integer r and an equilibrium point x^* of (1.1.1) such that $f^r(x) = x^*$, $f^{r-1}(x) \neq x^*$, then x is an *eventually equilibrium (fixed) point*.

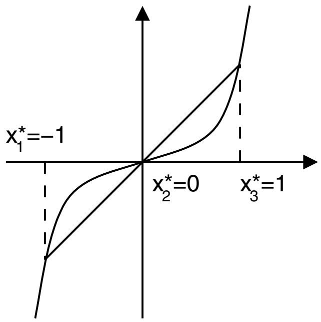


FIGURE 1.1. Fixed points of $f(x) = x^3$.

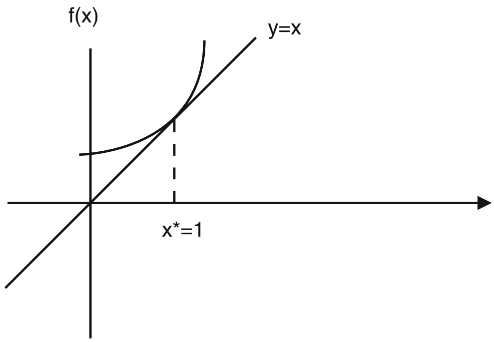


FIGURE 1.2. Fixed points of $f(x) = x^2 - x + 1$.

Example 1.7. The Tent Map

Consider the equation (Figure 1.3)

$$x(n+1) = T(x(n)),$$

where

$$T(x) = \begin{cases} 2x & \text{for } 0 \le x \le \dfrac{1}{2}, \\ 2(1-x) & \text{for } \dfrac{1}{2} < x \le 1. \end{cases}$$

There are two equilibrium points, 0 and $\frac{2}{3}$ (see Figure 1.3). The search for eventually equilibrium points is not as simple algebraically. If $x(0) = \frac{1}{4}$, then $x(1) = \frac{1}{2}$, $x(2) = 1$, and $x(3) = 0$. Thus $\frac{1}{4}$ is an eventually equilibrium point. The reader is asked to show that if $x = k/2^n$, where k and n are positive integers with $0 < k/2^n \le 1$, then x is an eventually equilibrium point (Exercises 1.3, Problem 15).

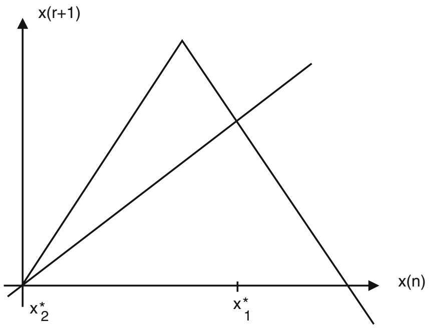


FIGURE 1.3. Equilibrium points of the tent map.

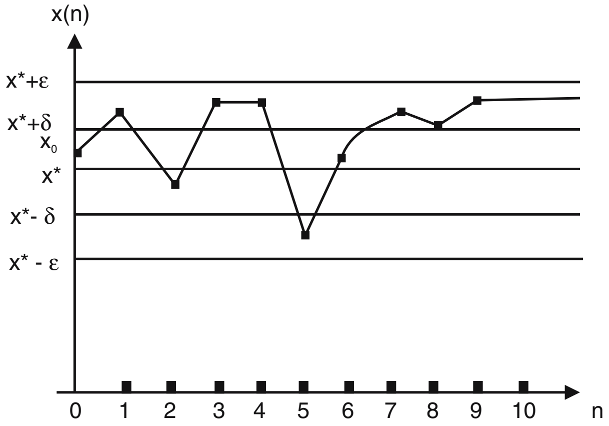


FIGURE 1.4. Stable x^*. If $x(0)$ is within δ from x^*, then $x(n)$ is within ε from $x(n)$ for all $n > 0$.

One of the main objectives in the study of a dynamical system is to analyze the behavior of its solutions near an equilibrium point. This study constitutes the stability theory. Next we introduce the basic definitions of stability.

Definition 1.8. (a) The equilibrium point x^* of (1.1.1) is *stable* (Figure 1.4) if given $\varepsilon > 0$ there exists $\delta > 0$ such that $|x_0 - x^*| < \delta$ implies $|f^n(x_0) - x^*| < \varepsilon$ for all $n > 0$. If x^* is not stable, then it is called *unstable* (Figure 1.5).

(b) The point x^* is said to be *attracting* if there exists $\eta > 0$ such that

$$|x(0) - x^*| < \eta \quad \text{implies} \quad \lim_{n\to\infty} x(n) = x^*.$$

If $\eta = \infty$, x^* is called a *global attractor* or *globally attracting*.

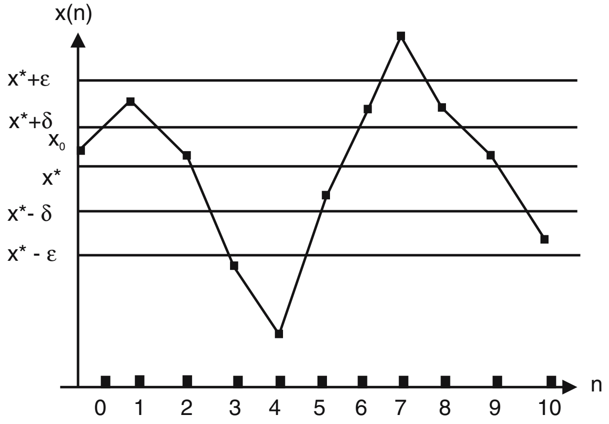


FIGURE 1.5. Unstable x^*. There exists $\varepsilon > 0$ such that no matter how close $x(0)$ is to x^*, there will be an N such that $x(N)$ is at least ε from x^*.

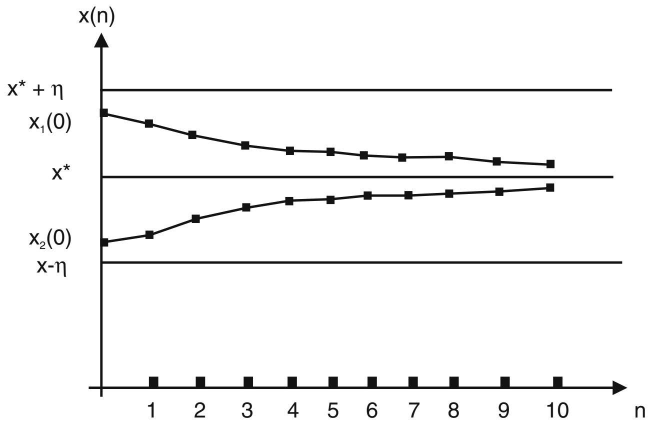


FIGURE 1.6. Asymptotically stable x^*. Stable if $x(0)$ is within η of x^*; then $\lim_{n\to\infty} x(n) = x^*$.

(c) The point x^* is an *asymptotically stable equilibrium point* if it is stable and attracting.

If $\eta = \infty$, x^* is said to be *globally asymptotically stable* (Figure 1.7).

To determine the stability of an equilibrium point from the above definitions may prove to be a mission impossible in many cases. This is due to the fact that we may not be able to find the solution in a closed form even for the deceptively simple-looking equation (1.1.1). In this section we present some of the simplest but most powerful tools of the trade to help us understand the behavior of solutions of (1.1.1) in the vicinity of equilib-

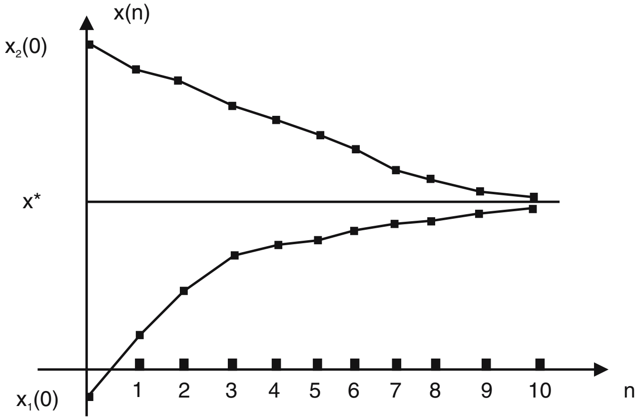


FIGURE 1.7. Globally asymptotically stable x^*. Stable and $\lim_{n\to\infty} x(n) = x^*$ for all $x(0)$.

rium points, namely, the graphical techniques. A hand-held calculator may fulfill all your graphical needs in this section.

1.3.1 The Stair Step (Cobweb) Diagrams

We now give, in excruciating detail, another important graphical method for analyzing the stability of equilibrium (and periodic) points for (1.1.1). Since $x(n+1) = f(x(n))$, we may draw a graph of f in the $(x(n), x(n+1))$ plane. Then, given $x(0) = x_0$, we pinpoint the value $x(1)$ by drawing a vertical line through x_0 so that it also intersects the graph of f at $(x_0, x(1))$. Next, draw a horizontal line from $(x_0, x(1))$ to meet the diagonal line $y = x$ at the point $(x(1), x(1))$. A vertical line drawn from the point $(x(1), x(1))$ will meet the graph of f at the point $(x(1), x(2))$. Continuing this process, one may find $x(n)$ for all $n > 0$.

Example 1.9. The Logistic Equation

Let $y(n)$ be the size of a population at time n. If μ is the rate of growth of the population from one generation to another, then we may consider a mathematical model in the form

$$y(n+1) = \mu y(n), \qquad \mu > 0. \tag{1.3.1}$$

If the initial population is given by $y(0) = y_0$, then by simple iteration we find that

$$y(n) = \mu^n y_0 \tag{1.3.2}$$

is the solution of (1.3.1). If $\mu > 1$, then $y(n)$ increases indefinitely, and $\lim_{n\to\infty} y(n) = \infty$. If $\mu = 1$, then $y(n) = y_0$ for all $n > 0$, which means that the size of the population is constant for the indefinite future. However, for $\mu < 1$, we have $\lim_{n\to\infty} y(n) = 0$, and the population eventually becomes extinct.

For most biological species, however, none of the above cases is valid as the population increases until it reaches a certain upper limit. Then, due to the limitations of available resources, the creatures will become testy and engage in competition for those limited resources. This competition is proportional to the number of squabbles among them, given by $y^2(n)$. A more reasonable model would allow b, the proportionality constant, to be greater than 0,

$$y(n+1) = \mu y(n) - by^2(n). \tag{1.3.3}$$

If in (1.3.3), we let $x(n) = \frac{b}{\mu} y(n)$, we obtain

$$x(n+1) = \mu x(n)(1 - x(n)) = f(x(n)). \tag{1.3.4}$$

This equation is the simplest nonlinear first-order difference equation, commonly referred to as the (discrete) logistic equation. However, a closed-form solution of (1.3.4) is not available (except for certain values of μ). In spite of its simplicity, this equation exhibits rather rich and complicated dynamics. To find the equilibrium points of (1.3.4) we let $f(x^*) = \mu x^*(1 - x^*) = x^*$. Thus, we pinpoint two equilibrium points: $x^* = 0$ and $x^* = (\mu - 1)/\mu$.

Figure 1.8 gives the stair step diagram of $(x(n), x(n+1))$ when $\mu = 2.5$ and $x(0) = 0.1$. In this case, we also have two equilibrium points. One, $x^* = 0$, is unstable, and the other, $x^* = 0.6$, is asymptotically stable.

Example 1.10. The Cobweb Phenomenon (Economics Application)

Here we study the pricing of a certain commodity. Let $S(n)$ be the number of units supplied in period n, $D(n)$ the number of units demanded in period n, and $p(n)$ the price per unit in period n.

For simplicity, we assume that $D(n)$ depends only linearly on $p(n)$ and is denoted by

$$D(n) = -m_d p(n) + b_d, \qquad m_d > 0, \qquad b_d > 0. \tag{1.3.5}$$

This equation is referred to as the price–demand curve. The constant m_d represents the sensitivity of consumers to price. We also assume that the price–supply curve relates the supply in any period to the price one period before, i.e.,

$$S(n+1) = m_s p(n) + b_s, \qquad m_s > 0, \qquad b_s > 0. \tag{1.3.6}$$

The constant m_s is the sensitivity of suppliers to price. The slope of the demand curve is negative because an increase of one unit in price produces a decrease of m_d units in demand. Correspondingly, an increase of one unit

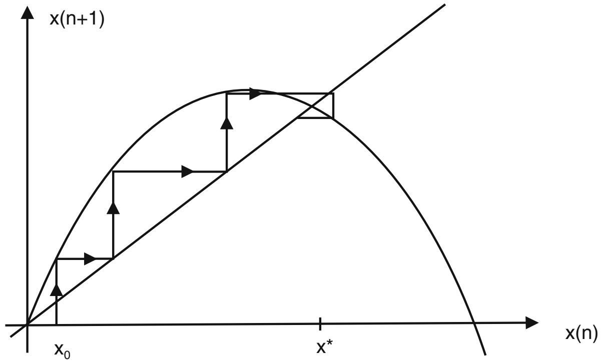


FIGURE 1.8. Stair step diagram for $\mu = 2.5$.

in price causes an increase of m_s units in supply, creating a positive slope for that curve.

A third assumption we make here is that the market price is the price at which the quantity demanded and the quantity supplied are equal, that is, at which $D(n+1) = S(n+1)$.

Thus

$$-m_d p(n+1) + b_d = m_s p(n) + b_s,$$

or

$$p(n+1) = Ap(n) + B = f(p(n)), \tag{1.3.7}$$

where

$$A = -\frac{m_s}{m_d}, \qquad B = \frac{b_d - b_s}{m_d}. \tag{1.3.8}$$

This equation is a first-order linear difference equation. The equilibrium price p^* is defined in economics as the price that results in an intersection of the supply $S(n+1)$ and demand $D(n)$ curves. Also, since p^* is the unique fixed point of $f(p)$ in (1.3.7), $p^* = B/(1-A)$. (This proof arises later as Exercises 1.3, Problem 6.) Because A is the ratio of the slopes of the supply and demand curves, this ratio determines the behavior of the price sequence. There are three cases to be considered:

(a) $-1 < A < 0$,

(b) $A = -1$,

(c) $A < -1$.

The three cases are now depicted graphically using our old standby, the stair step diagram.

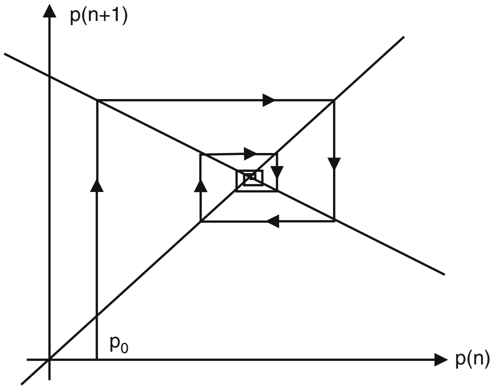


FIGURE 1.9. Asymptotically stable equilibrium price.

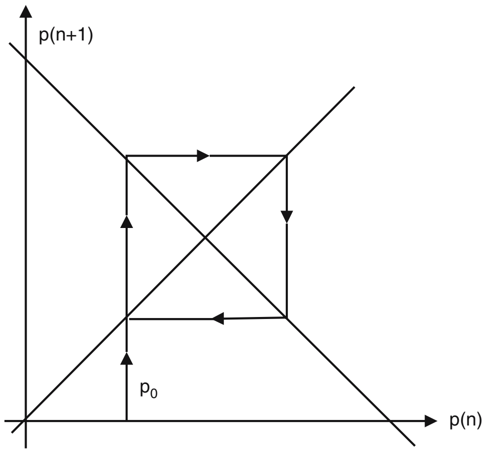


FIGURE 1.10. Stable equilibrium price.

(i) In case (a), prices alternate above and below but converge to the equilibrium price p^*. In economics lingo, the price p^* is considered "stable"; in mathematics, we refer to it as "asymptotically stable" (Figure 1.9).

(ii) In case (b), prices oscillate between two values only. If $p(0) = p_0$, then $p(1) = -p_0 + B$ and $p(2) = p_0$. Hence the equilibrium point p^* is stable (Figure 1.10).

(iii) In case (c), prices oscillate infinitely about the equilibrium point p^* but progressively move further away from it. Thus, the equilibrium point is considered unstable (Figure 1.11).

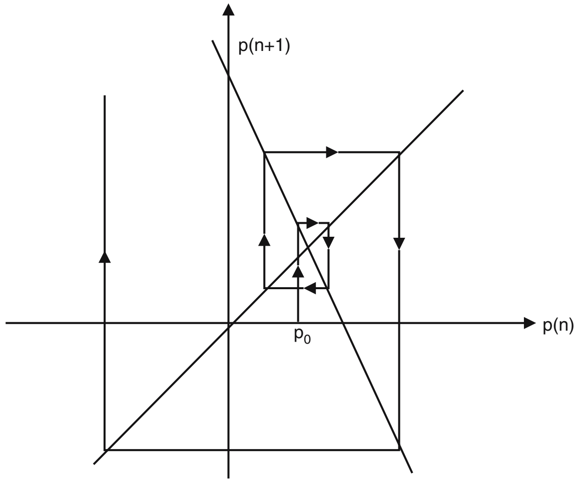


FIGURE 1.11. Unstable equilibrium price.

An explicit solution of (1.3.7) with $p(0) = p_0$ is given by

$$p(n) = \left(p_0 - \frac{B}{1-A}\right) A^n + \frac{B}{1-A} \text{ (Exercises 1.3, Problem 9).} \qquad (1.3.9)$$

This explicit solution allows us to restate cases (a) and (b) as follows.

1.3.2 The Cobweb Theorem of Economics

If the suppliers are less sensitive to price than the consumers (i.e., $m_s < m_d$), the market will then be stable. If the suppliers are more sensitive than the consumers, the market will be unstable.

One might also find the closed-form solution (1.3.9) by using a computer algebra program, such as Maple. One would enter this program:

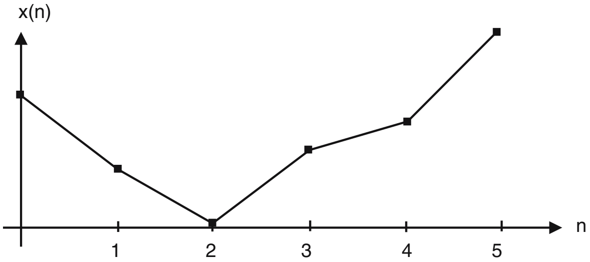
$$\text{rsolve}(\{p(n+1) = a * p(n) + b, p(0) = p_0\}, p(n)).$$

Exercises 1.3

1. Contemplate the equation $x(n+1) = f(x(n))$, where $f(0) = 0$.

   (a) Prove that $x(n) \equiv 0$ is a solution of the equation.

   (b) Show that the function depicted in the following $(n, x(n))$ diagram cannot possibly be a solution of the equation:



2. (Newton's Method of Computing the Square Root of a Positive Number)
The equation $x^2 = a$ can be written in the form $x = \frac{1}{2}(x + a/x)$. This form leads to Newton's method

$$x(n+1) = \frac{1}{2}\left[x(n) + \frac{a}{x(n)}\right].$$

   (a) Show that this difference equation has two equilibrium points, $-\sqrt{a}$ and $\sqrt{a}$.

   (b) Sketch a stair step diagram for $a = 3, x(0) = 1$, and $x(0) = -1$.

   (c) What can you conclude from (b)?

3. (Pielou's Logistic Equation)
E.C. Pielou [119] referred to the following equation as the discrete logistic equation:

$$x(n+1) = \frac{\alpha x(n)}{1 + \beta x(n)}, \qquad \alpha > 1, \qquad \beta > 0.$$

   (a) Find the positive equilibrium point.

   (b) Demonstrate, using the stair step diagram, that the positive equilibrium point is asymptotically stable, taking $\alpha = 2$ and $\beta = 1$.

4. Find the equilibrium points and determine their stability for the equation

$$x(n+1) = 5 - \frac{6}{x(n)}.$$

5. (a) Draw a stair step diagram for (1.3.4) for $\mu = 0.5$, 3, and 3.3. What can you conclude from these diagrams?

   (b) Determine whether these values for μ give rise to periodic solutions of period 2.

6. (The Cobweb Phenomenon [equation (1.3.7)]). Economists define the equilibrium price p^* of a commodity as the price at which the demand function $D(n)$ is equal to the supply function $S(n+1)$. These are defined in (1.3.5) and (1.3.6), respectively.

(a) Show that $p* = \frac{B}{1-A}$, where A and B are defined as in (1.3.8).

(b) Let $m_s = 2$, $b_s = 3$, $m_d = 1$, and $b_d = 15$. Find the equilibrium price $p*$. Then draw a stair step diagram, for $p(0) = 2$.

7. Continuation of Problem 6:
Economists use a different stair step diagram, as we will explain in the following steps:

(i) Let the x-axis represent the price $p(n)$ and the y-axis represent $S(n+1)$ or $D(n)$. Draw the supply line and the demand line and find their point of intersection $p*$.

(ii) Starting with $p(0) = 2$ we find $s(1)$ by moving vertically to the supply line, then moving horizontally to find $D(1)$ (since $D(1) = S(1)$), which determines $p(1)$ on the price axis. The supply $S(2)$ is found on the supply line directly above $p(1)$, and then $D(2)$ ($= S(2)$) is found by moving horizontally to the demand line, etc.

(iii) Is p^* stable?

8. Repeat Exercises 6 and 7 for:

(a) $m_s = m_d = 2, b_d = 10$, and $b_s = 2$.

(b) $m_s = 1, m_d = 2, b_d = 14$, and $b_s = 2$.

9. Verify that formula (1.3.9) is a solution of (1.3.7).

10. Use formula (1.3.9) to show that:

(a) If $-1 < A < 0$, then $\lim_{n\to\infty} p(n) = B/1 - A$.

(b) If $A < -1$, then $p(n)$ is unbounded.

(c) If $A = -1$, then $p(n)$ takes only two values:

$$p(n) = \begin{cases} p(0) & \text{if } n \text{ is even,} \\ p(1) = B - p_0 & \text{if } n \text{ is odd.} \end{cases}$$

11. Suppose that the supply and demand equations are given by $D(n) = -2p(n) + 3$ and $S(n+1) = p^2(n) + 1$.

(a) Assuming that the market price is the price at which supply equals demand, find a difference equation that relates $p(n+1)$ to $p(n)$.

(b) Find the positive equilibrium value of this equation.

(c) Use the stair step diagrams to determine the stability of the positive equilibrium value.

12. Consider Baker's map defined by

$$B(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq \dfrac{1}{2}, \\ 2x-1 & \text{for } \dfrac{1}{2} < x \leq 1. \end{cases}$$

    (i) Draw the function $B(x)$ on [0,1].

    (ii) Show that $x \in [0,1]$ is an eventually fixed point if and only if it is of the form $x = k/2^n$, where k and n are positive integers,[2] with $0 \leq k \leq 2^n - 1$.

13. Find the fixed points and the eventually fixed points of $x(n+1) = f(x(n))$, where $f(x) = x^2$.

14. Find an eventually fixed point of the tent map of Example 1.7 that is not in the form $k/2^n$.

15. Consider the tent map of Example 1.7. Show that if $x = k/2^n$, where k and n are positive integers with $0 < k/2^n \leq 1$, then x is an eventually fixed point.

1.4 Numerical Solutions of Differential Equations

Differential equations have been extensively used as mathematical models for a wide variety of physical and artificial phenomena. Such models describe populations or objects that evolve continuously in which time (or the independent variable) is a subset of the set of real numbers. In contrast, difference equations describe populations or objects that evolve discretely in which time (or the independent variable) is a subset of the set of integers. In many instances, one is unable to solve a given differential equation. In this case, we need to use a numerical scheme to approximate the solutions of the differential equations. A numerical scheme leads to the construction of an associated difference equation that is more amenable to computation either by a graphing-held calculator or by a computer. Here we present a couple of simple numerical schemes. We begin by *Euler's method*, one of the oldest numerical methods.

1.4.1 Euler's Method

Consider the first-order differential equation

$$x'(t) = g(t, x(t)), \qquad x(t_0) = x_0, \qquad t_0 \leq t \leq b. \tag{1.4.1}$$

[2] A number $x \in [0,1]$ is called a dyadic rational if it has the form $k/2^n$ for some nonnegative integers k and n, with $0 \leq k \leq 2^n - 1$.

Let us divide the interval $[t_0, b]$ into N equal subintervals. The size of each subinterval is called the *step size* of the method and is denoted by $h = (b - t_0)/N$. This step size defines the *nodes* $t_0, t_1, t_2, \ldots, t_N$, where $t_j = t_0 + jh$. Euler's method approximates $x'(t)$ by $(x(t+h) - x(t))/h$.

Substituting this value into (1.4.1) gives

$$x(t+h) = x(t) + hg(t, x(t)),$$

and for $t = t_0 + nh$, we obtain

$$x[t_0 + (n+1)h] = x(t_0 + nh) + hg[t_0 + nh, x(t_0 + nh)] \tag{1.4.2}$$

for $n = 0, 1, 2, \ldots, N-1$.

Adapting the difference equation notation and replacing $x(t_0 + nh)$ by $x(n)$ gives

$$x(n+1) = x(n) + hg[n, x(n)]. \tag{1.4.3}$$

Equation (1.4.3) defines *Euler's algorithm*, which approximates the solutions of the differential equation (1.4.1) at the node points.

Note that x^* is an equilibrium point of (1.4.3) if and only if $g(x^*) = 0$. Thus the differential equation (1.4.1) and the difference equation (1.4.3) have the same equilibrium points.

Example 1.11. Let us now apply Euler's method to the differential equation:

$$x'(t) = 0.7x^2(t) + 0.7, \qquad x(0) = 1, \qquad t \in [0, 1] \qquad (DE) \text{ (see footnote 3).}$$

Using the separation of variable method, we obtain

$$\frac{1}{0.7} \int \frac{dx}{x^2 + 1} = \int dt.$$

Hence

$$\tan^{-1}(x(t)) = 0.7t + c.$$

Letting $x(0) = 1$, we get $c = \frac{\pi}{4}$. Thus, the exact solution of this equation is given by $x(t) = \tan\left(0.7t + \frac{\pi}{4}\right)$.

The corresponding difference equation using Euler's method is

$$x(n+1) = x(n) + 0.7h(x^2(n) + 1), \qquad x(0) = 1 \qquad (\Delta E) \text{ (see footnote 4)}$$

Table 1.3 shows the Euler approximations for $h = 0.2$ and 0.1, as well as the exact values. Figure 1.12 depicts the $(n, x(n))$ diagram. Notice that the smaller the step size we use, the better the approximation we have.

[3] $DE \equiv$ differential equation.
[4] $\Delta E \equiv$ difference equation.

TABLE 1.3.

n	t	(ΔE) Euler ($h = 0.2$) $x(n)$	(ΔE) Euler ($h = 0.1$) $x(n)$	Exact (DE) $x(t)$
0	0	1	1	1
1	0.1		1.14	1.150
2	0.2	1.28	1.301	1.328
3	0.3		1.489	1.542
4	0.4	1.649	1.715	1.807
5	0.5		1.991	2.150
6	0.6	2.170	2.338	2.614
7	0.7		2.791	3.286
8	0.8	2.969	3.406	4.361
9	0.9		4.288	6.383
10	1	4.343	5.645	11.681

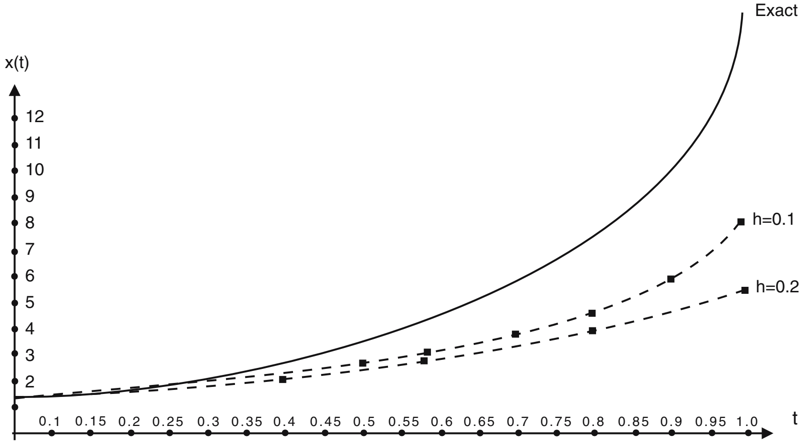


FIGURE 1.12. The $(n, x(n))$ diagram.

Example 1.12. Consider the logistic differential equation

$$x'(t) = ax(t)(1 - x(t)), \quad x(0) = x_0.$$

The equilibrium points (or constant solutions) are obtained by letting $x'(t) = 0$. Hence $ax(1 - x) = 0$ and we then have two equilibrium points $x_1^* = 0$ and $x_2^* = 1$. The exact solution of the equation is obtained by separation of variables,

$$\frac{dx}{x(1-x)} = a\,dt,$$
$$\int \frac{dx}{x} + \int \frac{dx}{1-x} = \int a\,dt,$$
$$\ln\left(\frac{x}{1-x}\right) = at + c,$$
$$\frac{x}{1-x} = e^{at+c} = be^{at}, \quad b = e^c.$$

Hence

$$x(t) = \frac{be^{at}}{1+be^{at}}.$$

Now $x(0) = x_0 = \frac{b}{1+b}$ gives $b = \frac{x_0}{1-x_0}$. Substituting in $x(t)$ yields

$$x(t) = \frac{x_0 e^{at}}{1 - x_0 + x_0 e^{at}} = \frac{x_0 e^{at}}{1 + x_0(e^{at}-1)}.$$

If $a > 0$, $\lim_{t\to\infty} x(t) = 1$, and thus all solutions converge to the equilibrium point $x_2^* = 1$. On the other hand, if $a < 0$, $\lim_{t\to\infty} x(t) = 0$, and thus all solutions converge to the equilibrium point $x_1^* = 0$.

Let us now apply Euler's method to the logistic differential equation. The corresponding difference equation is given by

$$x(n+1) = x(n) + h\,ax(n)(1-x(n)), \quad x(0) = x_0.$$

This equation has two equilibrium points $x_1^* = 0$, $x_2^* = 1$ as in the differential equation case.

Let $y(n) = \frac{ha}{1+ha}x(n)$. Then we have

$$y(n+1) = (1+ha)y(n)(1-y(n))$$

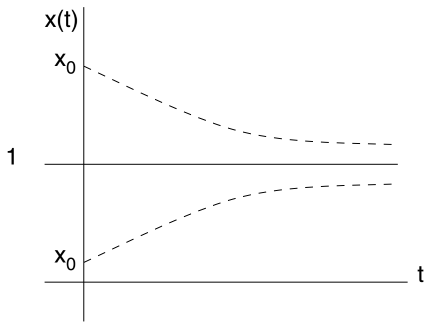


FIGURE 1.13. If $a > 0$, all solutions with $x_0 > 0$ converge to $x_2^* = 1$.

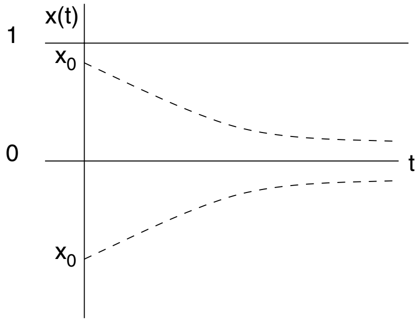


FIGURE 1.14. If $a < 0$, all solutions with $x_0 < 1$ converge to $x_1^* = 0$.

or

$$y(n+1) = \mu y(n)(1 - y(n)), \quad y(0) = \frac{ha}{1+ha}x(0), \quad \text{and} \quad \mu = 1 + ha.$$

The corresponding equilibrium points are $y_1^* = 0$ and $y_2^* = \frac{\mu-1}{\mu} = \frac{ha}{1+ha}$ which correspond to $x_1^* = 0$ and $x_2^* = 1$, respectively. Using the Cobweb diagram, we observe that for $1 < \mu < 3$ $(0 < ha < 2)$, all solutions whose initial point y_0 in the interval $(0, 1)$ converge to the equilibrium point $y_2^* = \frac{ha}{1+ha}$ (Figure 1.15) and for $0 < \mu < 1$ $(-1 < ha < 0)$, all solutions whose initial point y_0 in the interval $(0, 1)$ converge to the equilibrium point $y_2^* = 0$ (Figure 1.16). However, for $\mu > 3$ $(ha > 2)$, almost all solutions where initial points are in the interval $(0, 1)$ do not converge to either equilibrium point y_1^* or y_2^*. In fact, we will see in later sections that for $\mu > 3.57$ $(ha > 2.57)$, solutions of the difference equation behave in a "chaotic" manner (Figure 1.17). In the next section we will explore another numerical scheme that has been proven effective in a lot of cases [100].

1.4.2 A Nonstandard Scheme

Consider again the logistic differential equation. Now if we replace $x^2(n)$ in Euler's method by $x(n)x(n+1)$ we obtain

$$x(n+1) = x(n) + hax(n) - hax(n)x(n+1).$$

Simplifying we obtain the rational difference equation

$$x(n+1) = \frac{(1+ha)x(n)}{1+hax(n)}$$

or

$$x(n+1) = \frac{\alpha x(n)}{1+\beta x(n)}$$

with $\alpha = 1 + ha$, $\beta = \alpha - 1 = ha$.

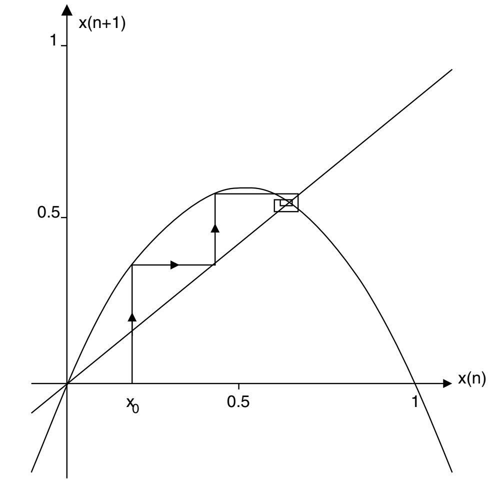


FIGURE 1.15. $0 < ha < 2$.

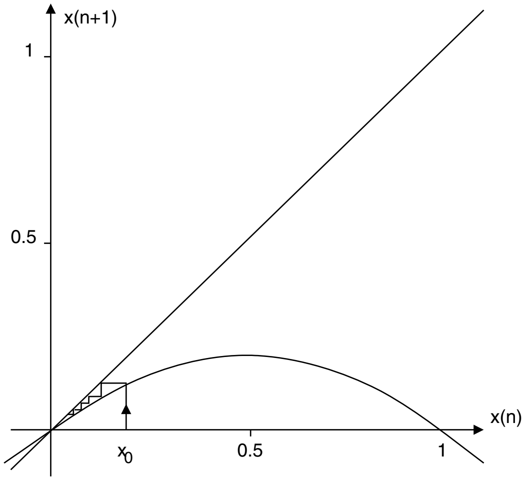


FIGURE 1.16. $-1 < ha < 0$.

This equation has two equilibrium points $x_1^* = 0$ and $x_2^* = 1$. From the Cobweb diagram (Figure 1.18) we conclude that $\lim_{n\to\infty} x(n) = 1$ if $\alpha > 1$.

Since $h > 0$, $\alpha > 1$ if and only if $a > 0$. Thus all solutions converge to the equilibrium point $x_2^* = 1$ if $a > 0$ as in the differential equation case regardless of the size of h.

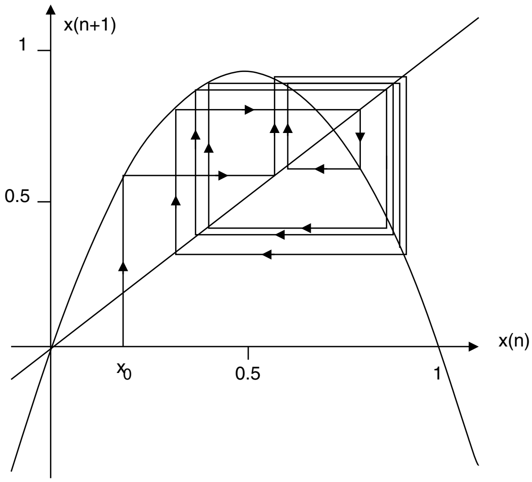


FIGURE 1.17. $ha > 2.57$.

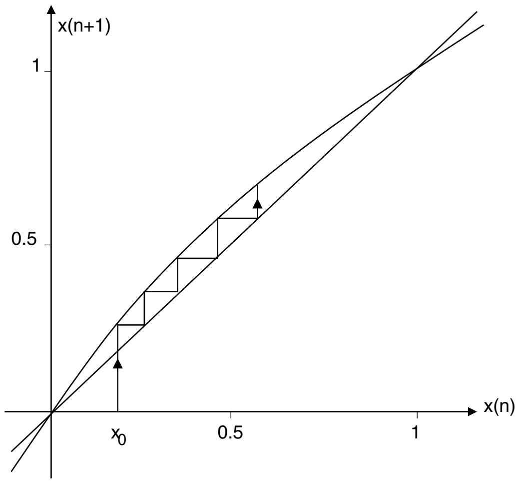


FIGURE 1.18. $\alpha = 1 + ha, \quad \beta = \alpha - 1 = ha$.

Exercises 1.4

In Problems 1–5

(a) Find the associated difference equation.

(b) Draw an $(n, y(n))$ diagram.

(c) Find, if possible, the exact solution of the differential equation and draw its graph on the same plot as that drawn in part in (b).

1. $y' = -y^2, \qquad y(0) = 1, \qquad 0 \le t \le 1, \qquad h = 0.2, 0.1.$
2. $y' = -y + \frac{4}{y}, \qquad y(0) = 1, \qquad 0 \le t \le 1, \qquad h = 0.25.$
3. $y' = -y + 1, \qquad y(0) = 2, \qquad 0 \le t \le 1, \qquad h = 0.25.$
4. $y' = y(1 - y), \qquad y(0) = 0.1, \qquad 0 \le t \le 1, \qquad h = 0.25.$
5. $y' = y^2 + 2, \qquad y(0) = \frac{1}{4}, \qquad 0 \le t \le 1, \qquad h = 0.25.$
6. Use a nonstandard numerical method to find the associated difference equation of the differential equation in Problem 1.
7. Do Problem 4 using a nonstandard numerical method and compare your results with Euler's method.
8. Do Problem 5 using a nonstandard numerical method and compare your result with Euler's method.
9. Use both Euler's method and a nonstandard method to discretize the differential equation

$$y'(t) = y^2 + t, \qquad y(0) = 1, \qquad 0 \le t \le 1, \qquad h = 0.2.$$

Draw the $n - y(n)$ diagram for both methods. Guess which method gives a better approximation to the differential equation.

10. (a) Use the Euler method with $h = 0.25$ on $[0, 1]$ to find the value of y corresponding to $t = 0.5$ for the differential equation

$$\frac{dy}{dt} = 2t + y, \qquad y(0) = 1.$$

(b) Compare the result obtained in (a) with the exact value.

11. Given the differential equation of Problem 10, show that a better approximation is given by the difference equation

$$y(n+1) = y(n) + \frac{1}{2}h(y'(n) + y'(n+1)).$$

This method is sometimes called the *modified Euler method.*

1.5 Criterion for the Asymptotic Stability of Equilibrium Points

In this section we give a simple but powerful criterion for the asymptotic stability of equilibrium points. The following theorem is our main tool in this section.

Theorem 1.13. *Let x^* be an equilibrium point of the difference equation*

$$x(n+1) = f(x(n)), \tag{1.5.1}$$

where f is continuously differentiable at x^*. The following statements then hold true:

(i) *If $|f'(x^*)| < 1$, then x^* is asymptotically stable.*

(ii) *If $|f'(x^*)| > 1$, then x^* is unstable.*

PROOF.

(i) Suppose that $|f'(x^*)| < M < 1$. Then there is an interval $J = (x^*-\gamma, x^*+\gamma)$ containing x^* such that $|f'(x)| \leq M < 1$ for all $x \in J$. For if not, then for each open interval $I_n = (x^* - \frac{1}{n}, x^* + \frac{1}{n})$ (for large n) there is a point $x_n \in I_n$ such that $|f'(x_n)| > M$. As $n \to \infty$, $x_n \to x^*$. Since f' is a continuous function, it follows that

$$\lim_{n\to\infty} f'(x_n) = f'(x^*).$$

Consequently,

$$M \leq \lim_{n\to\infty} |f'(x_n)| = |f'(x^*)| < M,$$

which is a contradiction. This proves our statement. For $x(0) \in J$, we have

$$|x(1) - x^*| = |f(x(0)) - f(x^*)|.$$

By the Mean Value Theorem, there exists ξ between $x(0)$ and x^* such that

$$|f(x(0)) - f(x^*)| = |f'(\xi)|\ |x(0) - x^*|.$$

Thus

$$|f(x(0)) - x^*| \leq M|x(0) - x^*|.$$

Hence

$$|x(1) - x^*| \leq M|x(0) - x^*|. \tag{1.5.2}$$

Since $M < 1$, inequality (1.5.2) shows that $x(1)$ is closer to x^* than $x(0)$. Consequently, $x(1) \in J$.

By induction we conclude that

$$|x(n) - x^*| \leq M^n|x(0) - x^*|.$$

For $\varepsilon > 0$ we let $\delta = \frac{\varepsilon}{2M}$. Thus $|x(0) - x^*| < \delta$ implies that $|x(n) - x^*| < \varepsilon$ for all $n > 0$. This conclusion suggests stability. Furthermore, $\lim_{n\to\infty} |x(n) - x^*| = 0$, and thus $\lim_{n\to\infty} x(n) = x^*$; we conclude asymptotic stability. □

The proof of part (ii) is left as Exercises 1.5, Problem 11.

Remark: In the literature of dynamical systems, the equilibrium point x^* is said to be *hyperbolic* if $|f'(x^*)| \neq 1$.

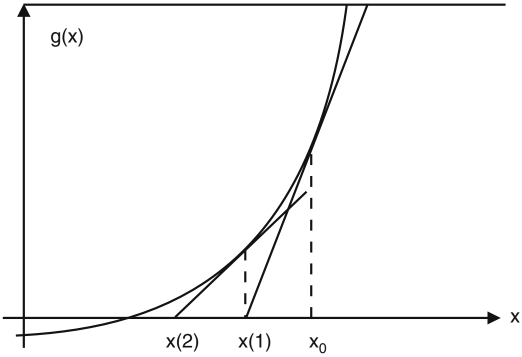


FIGURE 1.19. Newton's method.

Example 1.14. The Newton–Raphson Method

The Newton–Raphson method is one of the most famous numerical methods for finding the roots of the equation $g(x) = 0$, where $g(x)$ is continually differentiable (i.e., its derivative exists and is continuous).

Newton's algorithm for finding a zero x^* of $g(x)$ is given by the difference equation

$$\boxed{x(n+1) = x(n) - \frac{g(x(n))}{g'(x(n))},} \tag{1.5.3}$$

where $x(0) = x_0$ is your initial guess of the root x^*. Here $f(x) = x - \frac{g(x)}{g'(x)}$.

Note first that the zero x^* of $g(x)$ is also an equilibrium point of (1.5.3). To determine whether Newton's algorithm provides a sequence $\{x(n)\}$ that converges to x^* we use Theorem 1.13:

$$|f'(x^*)| = \left| 1 - \frac{[g'(x^*)]^2 - g(x^*)g''(x^*)}{[g'(x^*)]^2} \right| = 0,$$

since $g(x^*) = 0$. By Theorem 1.13, $\lim_{n\to\infty} x(n) = x^*$ if $x(0) = x_0$ is close enough to x^* and $g'(x^*) \neq 0$.

Observe that Theorem 1.13 does not address the nonhyperbolic case where $|f'(x^*)| = 1$. Further analysis is needed here to determine the stability of the equilibrium point x^*. Our first discussion will address the case where $f'(x^*) = 1$.

Theorem 1.15. *Suppose that for an equilibrium point x^* of* (1.5.1), $f'(x^*) = 1$. *The following statements then hold:*

(i) *If $f''(x^*) \neq 0$, then x^* is unstable.*

(ii) *If $f''(x^*) = 0$ and $f'''(x^*) > 0$, then x^* is unstable.*

(iii) *If $f''(x^*) = 0$ and $f'''(x^*) < 0$, then x^* is asymptotically stable.*

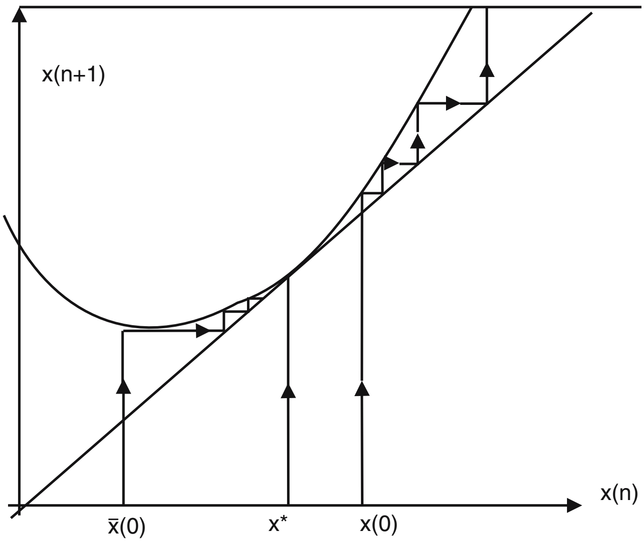


FIGURE 1.20. Unstable. $f''(x^*) > 0$ (semistable from the left).

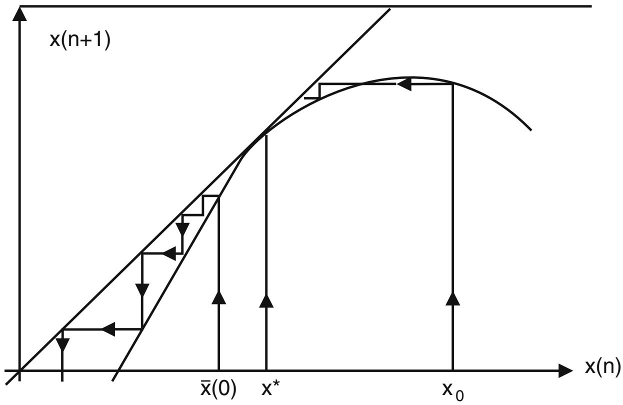


FIGURE 1.21. Unstable. $f''(x^*) < 0$ (semistable from the right).

PROOF.

(i) If $f''(x^*) \neq 0$, then the curve $y = f(x)$ is either concave upward if $f''(x^*) > 0$ or concave downward if $f''(x^*) < 0$, as shown in Figures 1.20, 1.21, 1.22, 1.23. If $f''(x^*) > 0$, then $f'(x) > 1$ for all x in a small interval $I = (x^*, x^* + \varepsilon)$. Using the same proof as in Theorem 1.13, it is easy to show that x^* is unstable. On the other hand, if $f''(x^*) < 0$, then $f'(x) > 1$ for all x in a small interval $I = (x^* - \varepsilon, x^*)$. Hence x^* is again unstable. □

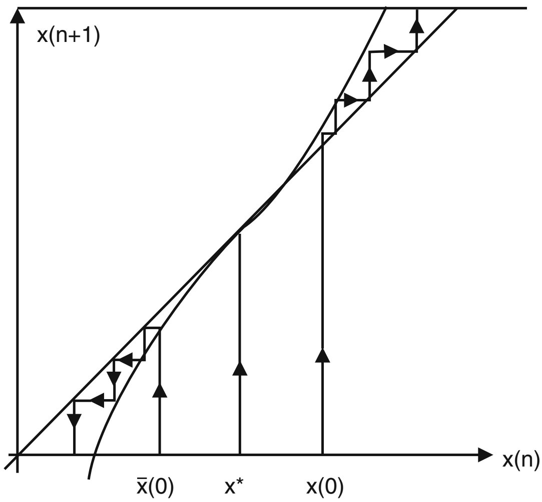


FIGURE 1.22. Unstable. $f'(x^*) = 1, f''(x^*) = 0$, and $f'''(x^*) > 0$.

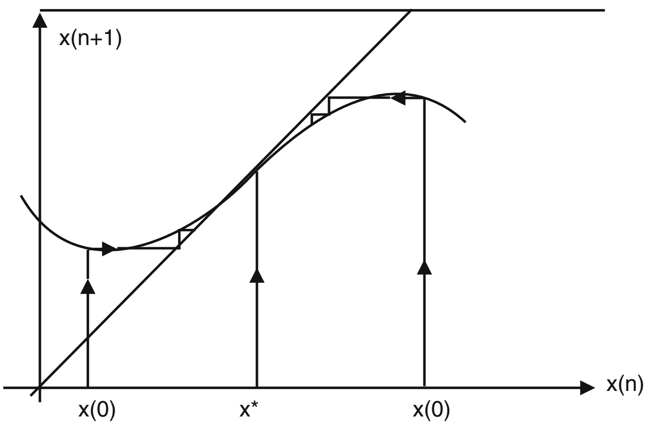


FIGURE 1.23. Asymptotically stable. $f'(x^*) = 1$, $f''(x^*) = 0$, and $f'''(x^*) < 0$.

Proofs of parts (ii) and (iii) remain for the student's pleasure as Exercises 1.5, Problem 14.

We now use the preceding result to investigate the case $f'(x^*) = -1$.

But before doing so, we need to introduce the notion of the Schwarzian derivative of a function f:

$$\boxed{Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2}$$

Note that if $f'(x^*) = -1$, then

$$\boxed{Sf(x^*) = -f'''(x^*) - \frac{3}{2}\left(f''(x^*)\right)^2.}$$

Theorem 1.16. *Suppose that for the equilibrium point x^* of* (1.1.1)*, $f'(x^*) = -1$. The following statements then hold:*

(i) *If $Sf(x^*) < 0$, then x^* is asymptotically stable.*

(ii) *If $Sf(x^*) > 0$, then x^* is unstable.*

PROOF. Contemplate the equation

$$y(n+1) = g(y(n)), \text{ where } g(y) = f^2(y). \tag{1.5.4}$$

We will make two observations about (1.5.4). First, the equilibrium point x^* of (1.1.1) is also an equilibrium point of (1.5.4). Second, if x^* is asymptotically stable (unstable) with respect to (1.5.4), then it is so with respect to (1.1.1). (Why?) (Exercises 1.5, Problem 12.) Now,

$$\frac{d}{dy}g(y) = \frac{d}{dy}f(f(y)) = f'(f(y))f'(y).$$

Thus $\frac{d}{dy}g(x^*) = [f'(x^*)]^2 = 1$. Hence Theorem 1.15 applies to this situation. We need to evaluate $\frac{d^2}{dy^2}g(x^*)$:

$$\begin{aligned}\frac{d^2}{dy^2}g(y) = \frac{d^2}{dy^2}f(f(y)) &= [f'(f(y))f'(y)]' \\ &= [f'(y)]^2 f''(f(y)) + f'(f(y))f''(y).\end{aligned}$$

Hence

$$\frac{d^2}{dy^2}g(x^*) = 0.$$

Now, Theorem 1.15 [parts (ii) and (iii)] tells us that the asymptotic stability of x^* is determined by the sign of $[g(x^*)]'''$. Using the chain rule again, one may show that

$$[g(x^*)]''' = -2f'''(x^*) - 3[f''(x^*)]^2. \tag{1.5.5}$$

(The explicit proof with the chain rule remains as Exercises 1.5, Problem 13.) This step rewards us with parts (i) and (ii), and the proof of the theorem is now complete. $\square$

Example 1.17. Consider the difference equation $x(n+1) = x^2(n) + 3x(n)$. Find the equilibrium points and determine their stability.
Solution The equilibrium points are 0 and -2. Now, $f'(x) = 2x + 3$. Since $f'(0) = 3$, it follows from Theorem 1.13 that 0 is unstable. Now, $f'(-2) = -1$, so Theorem 1.16 applies. Using (1.5.5) we obtain $-2f'''(-2) -$

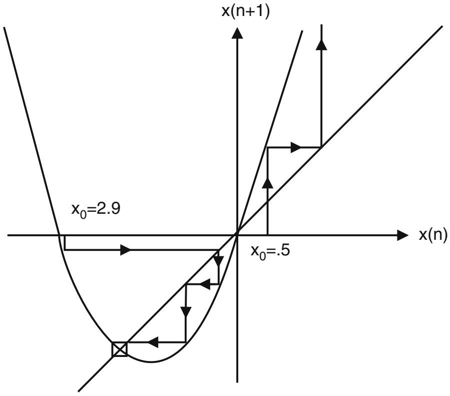


FIGURE 1.24. Stair step diagram for $x(n+1) = x^2(n) + 3x(n)$.

$3[f''(-2)]^2 = -12 < 0$. Theorem 1.16 then declares that the equilibrium point -2 is asymptotically stable. Figure 1.24 illustrates the stair step diagram of the equation.

Remark: One may generalize the result in the preceding example to a general quadratic map $Q(x) = ax^2 + bx + c, a \neq 0$. Let x^* be an equilibrium point of $Q(x)$, i.e., $Q(x^*) = x^*$. Then the following statements hold true.

(i) If $Q'(x^*) = -1$, then by Theorem 1.16, the equilibrium point x^* is asymptotically stable. In fact, there are two equilibrium points for $Q(x)$;

$$x_1^* = [(1-b) - \sqrt{(b-1)^2 - 4ac}]/2a;$$
$$x_2^* = [(1-b) + \sqrt{(b-1)^2 - 4ac}]/2a.$$

It is easy to see that $Q'(x_1^*) = -1$, if $(b-1)^2 = 4ac+4$ and $Q'(x_2^*) \neq -1$. Thus x_1^* is asymptotically stable if $(b-1)^2 = 4ac + 4$ (Exercises 1.5, Problem 8).

(ii) If $Q'(x^*) = 1$, then by Theorem 1.15, x^* is unstable. In this case, we have only one equilibrium point $x^* = (1-b)/2a$. Thus, x^* is unstable if $(b-1)^2 = 4ac$.

Remark:

(i) Theorem 1.15 fails if for a fixed point x^*, $f'(x^*) = 1$, $f''(x^*) = f'''(x^*) = 0$. For example, for the map $f(x) = x + (x-1)^4$ and its fixed point $x^* = 1$, $f'(x^*) = 1$, $f''(x^*) = f'''(x^*) = 0$, and $f^{(4)}(x^*) = 24 > 0$.

(ii) Theorem 1.16 fails if $f'(x^*) = -1$, and $Sf(x^*) = 0$. This may be illustrated by the function $f(x) = -x + 2x^2 - 4x^3$. For the fixed $x^* = 0$, $f'(x^*) = -1$, and $Sf(x^*) = 0$.

In Appendix A, we present the general theory developed by Dannan, Elaydi, and Ponomarenko in 2003 [30]. The stability of the fixed points in the above examples will be determined.

Exercises 1.5

In Problems 1 through 7, find the equilibrium points and determine their stability using Theorems 1.13, 1.15, and 1.16.

1. $x(n+1) = \frac{1}{2}[x^3(n) + x(n)]$.
2. $x(n+1) = x^2(n) + \frac{1}{8}$.
3. $x(n+1) = \tan^{-1} x(n)$.
4. $x(n+1) = x^2(n)$.
5. $x(n+1) = x^3(n) + x(n)$.
6. $x(n+1) = \dfrac{\alpha x(n)}{1+\beta x(n)}, \qquad \alpha > 1$ and $\beta > 0$.
7. $x(n+1) = -x^3(n) - x(n)$.
8. Let $Q(x) = ax^2 + bx + c$, $a \neq 0$, and let x^* be a fixed point of Q. Prove the following statements:
   (i) If $Q'(x^*) = -1$, then x^* is asymptotically stable. Then prove the rest of Remark (i).
   (ii) If $Q'(x^*) = 1$, then x^* is unstable. Then prove the rest of Remark (ii).
9. Suppose that in (1.5.3), $g(x^*) = g'(x^*) = 0$ and $g''(x^*) \neq 0$. Prove that x^* is an equilibrium point of (1.5.3).
10. Prove Theorem 1.13, part (ii).
11. Prove that if x^* is an equilibrium point of (1.5.1), then it is an equilibrium point of (1.5.1). Show also that the converse is false in general. For what class of maps $f(x)$ does the converse hold?
12. Prove that if an equilibrium point x^* of (1.5.1) is asymptotically stable with respect to (1.5.4) (or unstable, as the case may be), it is also so with respect to (1.1.1).
13. Verify formula (1.5.5).
14. Prove Theorem 1.15, parts (ii) and (iii).
15. Definition of *Semistability.* An equilibrium point x^* of $x(n+1) = f(x(n))$ is semistable (from the right) if given $\varepsilon > 0$ there exists $\delta > 0$ such that if $x(0) > x^*$, $x(0) - x^* < \delta$, then $x(n) - x^* < \varepsilon$. Semistability from the left is defined similarly. If in addition, $\lim_{n\to\infty} x(n) = x^*$

whenever $x(0) - x^* < \eta\{x^* - x(0) < \eta\}$, then x^* is said to be semiasymptotically stable from the right {or from the left, whatever the case may be}.
Suppose that if $f'(x^*) = 1$, then $f''(x^*) \neq 0$. Prove that x^* is:

(i) semiasymptotically stable from the right from the right if $f''(x^*) < 0$;

(ii) semiasymptotically stable from the left from the left if $f''(x^*) > 0$.

16. Determine whether the equilibrium point $x^* = 0$ is semiasymptotically stable from the left or from the right.

(a) $x(n+1) = x^3(n) + x^2(n) + x(n)$.

(b) $x(n+1) = x^3(n) - x^2(n) + x(n)$.

1.6 Periodic Points and Cycles

The second most important notion in the study of dynamical systems is the notion of periodicity. For example, the motion of a pendulum is periodic. We have seen in Example 1.10 that if the sensitivity m_s of the suppliers to price is equal to the sensitivity of consumers to price, then prices oscillate between two values only.

Definition 1.18. Let b be in the domain of f. Then:

(i) b is called a *periodic* point of f (or of (1.5.1)) if for some positive integer k, $f^k(b) = b$. Hence a point is *k-periodic* if it is a fixed point of f^k, that is, if it is an equilibrium point of the difference equation

$$x(n+1) = g(x(n)), \tag{1.6.1}$$

where $g = f^k$.

The periodic orbit of b, $O(b) = \{b, f(b), f^2(b), \ldots, f^{k-1}(b)\}$, is often called a k-cycle.

(ii) b is called *eventually k-periodic* if for some positive integer m, $f^m(b)$ is a *k-periodic* point. In other words, b is eventually k-periodic if

$$f^{m+k}(b) = f^m(b).$$

Graphically, a k-periodic point is the x-coordinate of the point where the graph of f^k meets the diagonal line $y = x$. Figure 1.25 depicts the graph of f^2, where f is the logistic map, which shows that there are four fixed points of f^2, of which two are fixed points of f as shown in Figure 1.26. Hence the other two fixed points of f^2 form a 2-cycle. Notice also that the point $x_0 = 0.3$ (in Figure 1.26) goes into a 2-cycle, and thus it is an eventually

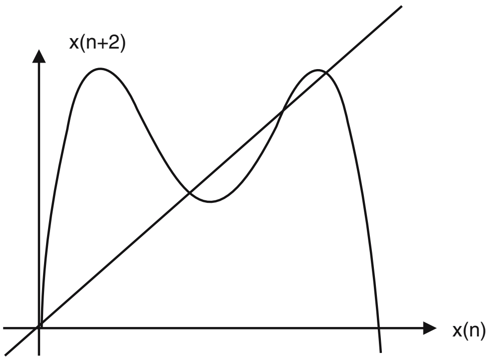


FIGURE 1.25. Graph of f^2 with four fixed points. $f(x) = 3.43x(1-x)$.

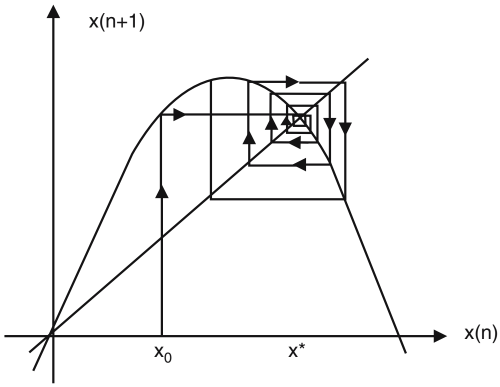


FIGURE 1.26. x_0 goes into a 2-cycle. $f(x) = 3.43x(1-x)$.

2-periodic point. Moreover, the point $x^* = 0.445$ is asymptotically stable relative to f^2 (Figure 1.27).

Observe also that if $A = -1$ in (1.3.7), then $f^2(p_0) = -(-p_0 + B) + B = p_0$. Therefore, every point is 2-periodic (see Figure 1.10). This means that in this case, if the initial price per unit of a certain commodity is p_0, then the price oscillates between p_0 and $B - p_0$.

Example 1.19. Consider again the difference equation generated by the tent function

$$T(x) = \begin{cases} 2x & \text{for } 0 \le x \le \dfrac{1}{2}, \\ 2(1-x) & \text{for } \dfrac{1}{2} < x \le 1 \ . \end{cases}$$

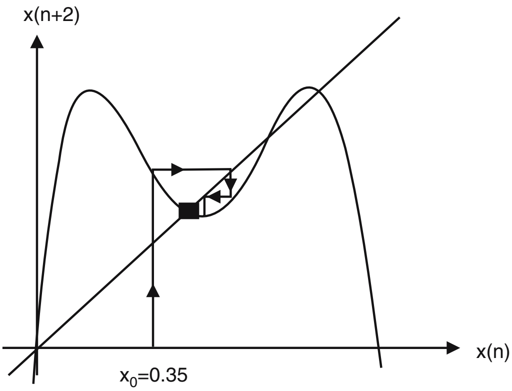


FIGURE 1.27. $x^* \approx 0.445$ is asymptotically stable relative to f^2.

This may also be written in the compact form

$$T(x) = 1 - 2\left|x - \frac{1}{2}\right|.$$

We first observe that the periodic points of period 2 are the fixed points of T^2. It is easy to verify that T^2 is given by

$$T^2(x) = \begin{cases} 4x & \text{for } 0 \le x < \dfrac{1}{4}, \\ 2(1-2x) & \text{for } \dfrac{1}{4} \le x < \dfrac{1}{2}, \\ 4\left(x - \dfrac{1}{2}\right) & \text{for } \dfrac{1}{2} \le x < \dfrac{3}{4}, \\ 4(1-x) & \text{for } \dfrac{3}{4} \le x \le 1. \end{cases}$$

There are four equilibrium points (Figure 1.28): 0, 0.4, $\frac{2}{3}$, and 0.8, two of which, 0 and $\frac{2}{3}$, are equilibrium points of T. Hence $\{0.4, 0.8\}$ is the only 2-cycle of T. Notice from Figure 1.29 that $x^* = 0.8$ is not stable relative to T^2.

Figure 1.30 depicts the graph of T^3. It is easy to verify that $\left\{\frac{2}{7}, \frac{4}{7}, \frac{6}{7}\right\}$ is a 3-cycle. Now,

$$T\left(\frac{2}{7}\right) = \frac{4}{7}, \qquad T\left(\frac{4}{7}\right) = \frac{6}{7}, \qquad T\left(\frac{6}{7}\right) = \frac{2}{7}.$$

Using a computer or hand-held calculator, one may show (using the stair step diagram) that the tent map T has periodic points of all periods. This is a phenomenon shared by all equations that possess a 3-cycle. It was discovered by Li and Yorke [92] in their celebrated paper "Period Three Implies Chaos."

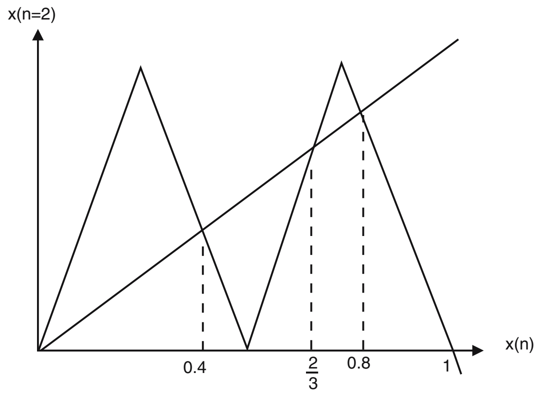


FIGURE 1.28. Fixed points of T^2.

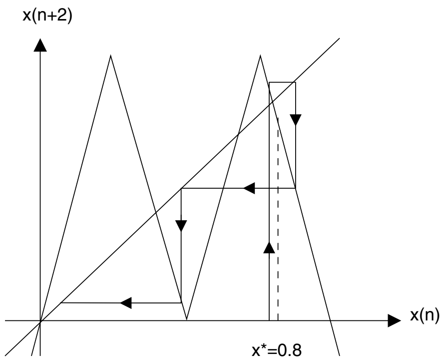


FIGURE 1.29. $x^* = 0.8$ is unstable relative to T^2.

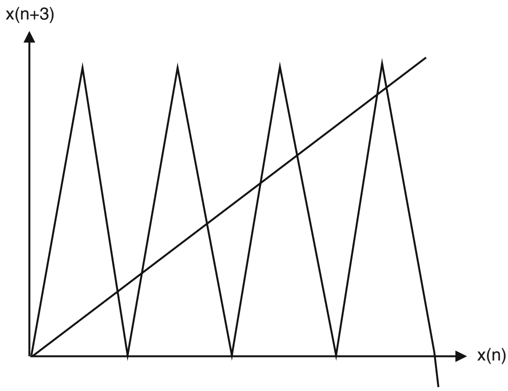


FIGURE 1.30. Fixed points of T^3.

We now turn our attention to explore the stability of periodic points.

Definition 1.20. Let b be a k-period point of f. Then b is:

(i) stable if it is a stable fixed point of f^k,

(ii) asymptotically stable if it is an asymptotically stable fixed point of f^k,

(iii) unstable if it is an unstable fixed point of f^k.

Notice that if b possesses a stability property, then so does every point in its k-cycle $\{x(0) = b, x(1) = f(b), x(2) = f^2(b), \ldots, x(k-1) = f^{k-1}(b)\}$. Hence we often speak of the stability of a k-cycle or a periodic orbit. Figure 1.29 shows that the 2-cycle in the tent map is not stable, since $x^* = 0.8$ is not stable as a fixed point of T^2, while the 2-cycle in the logistic map is asymptotically stable (see Figure 1.27).

Since the stability of a k-periodic point b of (1.1.1) reduces to the study of the stability of the point as an equilibrium point of (1.6.1), one can use all the theorems in the previous section applied to f^k. For example, Theorem 1.13 may be modified as follows.

Theorem 1.21. *Let $O(b) = \{b = x(0), x(1), \ldots, x(k-1)\}$ be a k-cycle of a continuously differentiable function f. Then the following statements hold:*

(i) *The k-cycle $O(b)$ is asymptotically stable if*

$$|f'(x(0))f'(x(1)), \ldots, f'(x(k-1))| < 1.$$

(ii) *The k-cycle $O(b)$ is unstable if*

$$|f'(x(0))f'(x(1)), \ldots, f'(x(k-1))| > 1.$$

Proof. We apply Theorem 1.13 to (1.6.1). Notice that by using the chain rule one may show that

$$[f^k(x(r))]' = f'(x(0))f'(x(1)), \ldots, f'(x(k-1)).$$

(See Exercises 1.6, Problem 12.) □

The conclusion of the theorem now follows.

Example 1.22. Consider the map $Q(x) = x^2 - 0.85$ defined on the interval $[-2, 2]$. Find the 2-cycles and determine their stability.
Solution Observe that $Q^2(x) = (x^2 - 0.85)^2 - 0.85$. The 2-periodic points are obtained by solving the equation

$$Q^2(x) = x, \text{ or } x^4 - 1.7x^2 - x - 0.1275 = 0. \tag{1.6.2}$$

This equation has four roots, two of which are fixed points of the map $Q(x)$. These two fixed points are the roots of the equation

$$x^2 - x - 0.85 = 0. \tag{1.6.3}$$

To eliminate these fixed points of $Q(x)$ from (1.6.2) we divide the left-hand side of (1.6.2) by the left-hand side of (1.6.3) to obtain the second-degree equation

$$x^2 + x + 0.15 = 0. \tag{1.6.4}$$

The 2-periodic points are now obtained by solving (1.6.4). They are given by

$$a = \frac{-1+\sqrt{0.4}}{2}, \qquad b = \frac{-1-\sqrt{0.4}}{2}.$$

To check the stability of the cycle $\{a, b\}$ we apply Theorem 1.21. Now,

$$|Q'(a)Q'(b)| = |(-1+\sqrt{0.4})(-1-\sqrt{0.4})| = 0.6 < 1.$$

Hence by Theorem 1.21, part (i), the 2-cycle is asymptotically stable.

Exercises 1.6

1. Suppose that the difference equation $x(n+1) = f(x(n))$ has a 2-cycle whose orbit is $\{a, b\}$. Prove that:

   (i) the 2-cycle is asymptotically stable if $|f'(a)f'(b)| < 1$,

   (ii) the 2-cycle is unstable if $|f'(a)f'(b)| > 1$.

2. Let T be the tent map in Example 1.17. Show that $\left\{\frac{2}{9}, \frac{4}{9}, \frac{8}{9}\right\}$ is an unstable 3-cycle for T.

3. Let $f(x) = -\frac{1}{2}x^2 - x + \frac{1}{2}$. Show that 1 is an asymptotically stable 2-periodic point of f.

In Problems 4 through 6 find the 2-cycle and then determine its stability.

4. $x(n+1) = 3.5x(n)[1 - x(n)]$.

5. $x(n+1) = 1 - x^2$.

6. $x(n+1) = 5 - (6/x(n))$.

7. Let $f(x) = ax^3 - bx + 1$, where $a, b \in \mathbb{R}$. Find the values of a and b for which $\{0, 1\}$ is an attracting 2-cycle.

   Consider Baker's function defined as follows:

$$B(x) = \begin{cases} 2x & \text{for } 0 \le x \le \dfrac{1}{2}, \\ 2x - 1 & \text{for } \dfrac{1}{2} < x \le 1. \end{cases}$$

Problems 8, 9, and 10 are concerned with Baker's function $B(x)$ on $[0, 1]$.

*8. (Hard). Draw Baker's function $B(x)$. Then find the number of n-periodic points of B.

9. Sketch the graph of B^2 and then find the 2-cycles of Baker's function B.

10. (Hard). Show that if m is an odd positive integer, then $\overline{x} = k/m$ is periodic, for $k = 1, 2, \ldots, m-1$.

11. Consider the quadratic map
$$Q(x) = ax^2 + bx + c, \quad a \neq 0.$$
   (a) If $\{d, e\}$ is a 2-cycle such that $Q'(d)Q'(e) = -1$, prove that it is asymptotically stable.

   (b) If $\{d, e\}$ is a 2-cycle with $Q'(d)Q'(e) = 1$, what can you say about the stability of the cycle?

12. (This exercise generalizes the result in Problem 1.) Let $\{x(0), x(1), \ldots, x(k-1)\}$ be a k-cycle of (1.2.1). Prove that:

   (i) if $|f'(x(0))f'(x(1)), \ldots, f'(x(k-1))| < 1$, then the k-cycle is asymptotically stable,

   (ii) if $|f'(x(0))f'(x(1)), \ldots, f'(x(k-1))| > 1$, then the k-cycle is unstable.

13. Give an example of a decreasing function that has a fixed point and a 2-cycle.

14. (i) Can a decreasing map have a k-cycle for $k > 1$?

   (ii) Can an increasing map have a k-cycle for $k > 1$?

Carvalho's Lemma. In [18] Carvalho gave a method to find periodic points of a given function. The method is based on the following lemma.

Lemma 1.23. *If k is a positive integer and $x(n)$ is a periodic sequence of period k, then the following hold true:*

(i) *If $k > 1$ is odd and $m = \frac{k-1}{2}$, then*
$$\boxed{x(n) = c_0 + \sum_{j=1}^{m} \left[c_j \cos\left(\frac{2jn\pi}{k}\right) + d_j \sin\left(\frac{2jn\pi}{k}\right)\right],}$$
for all $n \geq 1$.

(ii) *If k is even and $k = 2m$, then*
$$\boxed{x(n) = c_0 + (-1)^n c_m + \sum_{j=1}^{m-1} \left[c_j \cos\left(\frac{2jn\pi}{k}\right) + d_j \sin\left(\frac{2jn\pi}{k}\right)\right],}$$
for all $n \geq 1$.

Example 1.24 [23]. Consider the equation

$$x(n+1) = x(n)\exp(r(1-x(n))), \tag{1.6.5}$$

which describes a population with a propensity to simple exponential growth at low densities and a tendency to decrease at high densities. The quantity $\lambda = \exp(r(1-x(n)))$ could be considered the density-dependent reproductive rate of the population. This model is plausible for a single-species population that is regulated by an epidemic disease at high density.

The nontrivial fixed point of this equation is given by $x^* = 1$. Now, $f'(1) = 1 - r$. Hence $x^* = 1$ is asymptotically stable if $0 < r \leq 2$ (check $r = 2$). At $r = 2, x^* = 1$ loses its stability and gives rise to an asymptotically stable 2-cycle. Carvalho's lemma implies

$$x(n) = a + (-1)^n b.$$

Plugging this into equation (1.6.5) yields

$$a - (-1)^n b = (a + (-1)^n b)\ \exp r(1 - a - (-1)^n b).$$

The shift $n \mapsto n+1$ gives

$$a + (-1)^n b = (a - (-1)^n b)\ \exp r(1 - a + (-1)^n b).$$

Hence

$$a^2 - b^2 = (a^2 - b^2)\ \exp 2r(1-a).$$

Thus either $a^2 = b^2$, which gives the trivial solution 0, or $a = 1$. Hence a 2-periodic solution has the form $x(n) = 1 + (-1)^n b$. Plugging this again into equation (1.6.5) yields

$$1 - (-1)^n b = (1 + (-1)^n b)\exp((-1)^{n+1} rb).$$

Let $y = (-1)^{n+1} b$.Then

$$1 + y = (1-y)e^{ry},$$
$$r = \frac{1}{y}\ln\left(\frac{1+y}{1-y}\right) = g(y).$$

The function g has its minimum at 0, where $g(0) = 2$. Thus, for $r < 2, g(y) = r$ has no solution, and we have no periodic points, as predicted earlier. However, each $r > 2$ determines values $\pm y_r$ and the corresponding coefficient $(-1)^n b$. Further analysis may show that this map undergoes bifurcation similar to that of the logistic map.

Exercises 1.6 (continued).

In Problems 15 through 20, use Carvalho's lemma (Lemma 1.23).

15. Consider Ricker's equation

$$x(n+1) = x(n)\exp(r(1 - x(n))).$$

Find the 2-period solution when $r > 2$.

16. The population of a certain species is modeled by the difference equation $x(n+1) = \mu x(n)e^{-x(n)}$, $x(n) \geq 0$, $\mu > 0$. For what values of μ does the equation have a 2-cycle?

17. Use Carvalho's lemma to find the values of c for which the map

$$Q_c(x) = x^2 + c, \quad c \in [-2, 0],$$

has a 3-cycle and then determine its stability.

18*. (Term Project). Find the values of μ where the logistic equation $x(n+1) = \mu x(n)[1 - x(n)]$ has a 3-periodic solution.

19. Use Carvalho's lemma to find the values of μ where the logistic equation $x(n+1) = \mu x(n)[1 - x(n)]$ has a 2-periodic solution.

20. Find the 3-periodic solutions of the equation $x(n+1) = ax(n)$, $a \neq 1$.

1.7 The Logistic Equation and Bifurcation

Let us now return to the most important example in this chapter: the logistic difference equation

$$x(n+1) = \mu x(n)[1 - x(n)], \tag{1.7.1}$$

which arises from iterating the function

$$F_\mu(x) = \mu x(1 - x), \qquad x \in [0, 1], \qquad \mu > 0. \tag{1.7.2}$$

1.7.1 Equilibrium Points

To find the equilibrium points (fixed points of F_μ) of (1.7.1) we solve the equation

$$F_\mu(x^*) = x^*.$$

Hence the fixed points are $0, x^* = (\mu - 1)/\mu$. Next we investigate the stability of each equilibrium point separately.

(a) The equilibrium point 0. (See Figures 1.31, 1.32.) Since $F'_\mu(0) = \mu$, it follows from Theorems 1.13 and 1.15 that:

  (i) 0 is an asymptotically stable fixed point for $0 < \mu < 1$,

  (ii) 0 is an unstable fixed point for $\mu > 1$.

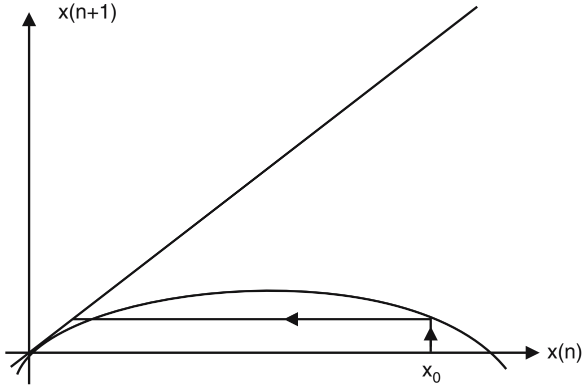


FIGURE 1.31. $0 < \mu < 1 : 0$ is an asymptotically stable fixed point.

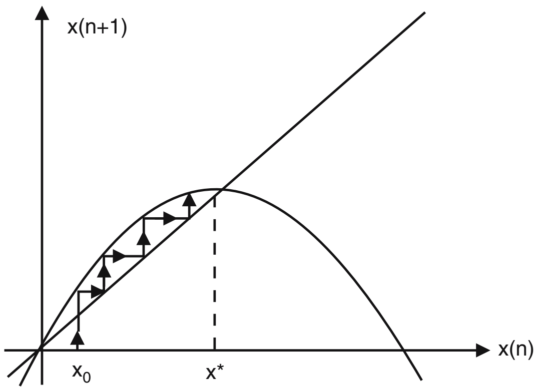


FIGURE 1.32. $\mu > 1 : 0$ is an unstable fixed point, x^* is an asymptotically fixed point.

The case where $\mu = 1$ needs special attention, for we have $F_1'(0) = 1$ and $F''(0) = -2 \neq 0$. By applying Theorem 1.15 we may conclude that 0 is unstable. This is certainly true if we consider negative as well as positive initial points in the neighborhood of 0. Since negative initial points are not in the domain of F_μ, we may discard them and consider only positive initial points. Exercises 1.5, Problem 16 tells us that 0 is semiasymptotically stable from the right, i.e., $x^* = 0$ is asymptotically stable in the domain $[0, 1]$.

(b) The equilibrium point $x^* = (\mu - 1)/\mu, \mu \neq 1$. (See Figures 1.32, 1.33.)

In order to have $x^* \in (0, 1]$ we require that $\mu > 1$. Now, $F_\mu'((\mu - 1)/\mu) = 2 - \mu$. Thus using Theorems 1.13 and 1.16 we obtain the following conclusions:

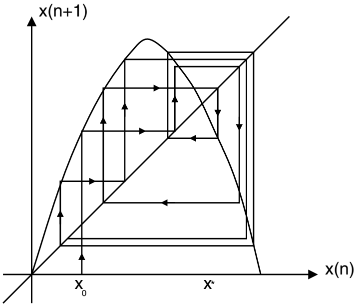


FIGURE 1.33. $\mu > 3$: x^* is an unstable fixed point.

(i) x^* is an asymptotically stable fixed point for $1 < \mu \leq 3$ (Figure 1.32).

(ii) x^* is an unstable fixed point for $\mu > 3$ (Figure 1.33).

1.7.2 2-Cycles

To find the 2-cycles we solve the equation $F_\mu^2(x) = x$ (or we solve $x_2 = \mu x_1(1-x_1), x_1 = \mu x_2(1-x_2)$),

$$\mu^2 x(1-x)[1-\mu x(1-x)] - x = 0. \tag{1.7.3}$$

Discarding the equilibrium points 0 and $x^* = \frac{\mu-1}{\mu}$, one may then divide (1.7.3) by the factor $x(x-(\mu-1)/\mu)$ to obtain the quadratic equation

$$\mu^2 x^2 - \mu(\mu+1)x + \mu + 1 = 0.$$

Solving this equation produces the 2-cycle

$$\begin{aligned} x(0) &= \left[(1+\mu) - \sqrt{(\mu-3)(\mu+1)}\right]/2\mu, \\ x(1) &= \left[(1+\mu) + \sqrt{(\mu-3)(\mu+1)}\right]/2\mu. \end{aligned} \tag{1.7.4}$$

Clearly, there are no periodic points of period 2 for $0 < \mu \leq 3$, and there is a 2-cycle for $\mu > 3$. For our reference we let $\mu_0 = 3$.

1.7.2.1 Stability of the 2-Cycle $\{x(0), x(1)\}$ for $\mu > 3$

From Theorem 1.21, this 2-cycle is asymptotically stable if

$$|F_\mu'(x(0))F_\mu'(x(1))| < 1,$$

or

$$-1 < \mu^2(1-2x(0))(1-2x(1)) < 1. \tag{1.7.5}$$

Substituting from (1.7.4) the values of $x(0)$ and $x(1)$ into (1.7.5), we obtain

$$3 < \mu < 1 + \sqrt{6} \approx 3.44949.$$

Conclusion This 2-cycle is attracting if $3 < \mu < 3.44949\ldots$.

Question What happens when $\mu = 1 + \sqrt{6}$?

In this case,

$$\left[F_\mu^2(x(0))\right]' = F_\mu'(x(0))F_\mu'(x(1)) = -1. \tag{1.7.6}$$

(Verify in Exercises 1.7, Problem 7.)

Hence we may use Theorem 1.16, part (i), to conclude that the 2-cycle is also attracting. For later reference, let $\mu_1 = 1 + \sqrt{6}$. Moreover, the 2-cycle becomes unstable when $\mu > \mu_1 = 1 + \sqrt{6}$.

1.7.3 2^2-*Cycles*

To find the 4-cycles we solve $F_\mu^4(x) = x$. The computation now becomes unbearable, and one should resort to a computer to do the work. It turns out that there is a 2^2-cycle when $\mu > 1 + \sqrt{6}$, which is attracting for $1 + \sqrt{6} < \mu < 3.544090\ldots$. This 2^2-cycle becomes unstable at $\mu > \mu_2 = 3.544090\ldots$.

When $\mu = \mu_2$, the 2^2-cycle bifurcates into a 2^3 cycle. The new 2^3 cycle is attracting for $\mu_3 < \mu \leq \mu_4$ for some number μ_4. This process of double bifurcation continues indefinitely. Thus we have a sequence $\{\mu_n\}_{n=0}^\infty$ where at μ_n there is a bifurcation from a 2^{n-1}-cycle to a 2^n-cycle. (See Figures 1.34, 1.35.) Table 1.4 provides some astonishing patterns.

From Table 1.4 we bring forward the following observations:

(i) The sequence $\{\mu_n\}$ seems to converge to a number $\mu_\infty = 3.57\ldots$.

(ii) The quotient $(\mu_n - \mu_{n-1})/(\mu_{n+1} - \mu_n)$ seems to tend to a number $\delta = 4.6692016\ldots$. This number is called the *Feigenbaum number* after its discoverer, the physicist Mitchell Feigenbaum [56]. In fact, Feigenbaum made a much more remarkable discovery: The number δ is universal and is independent of the form of the family of maps f_μ. However, the number μ_∞ depends on the family of functions under consideration.

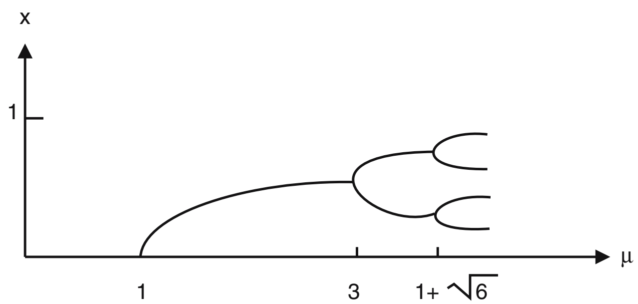


FIGURE 1.34. Partial bifurcation diagram for $\{F_\mu\}$.

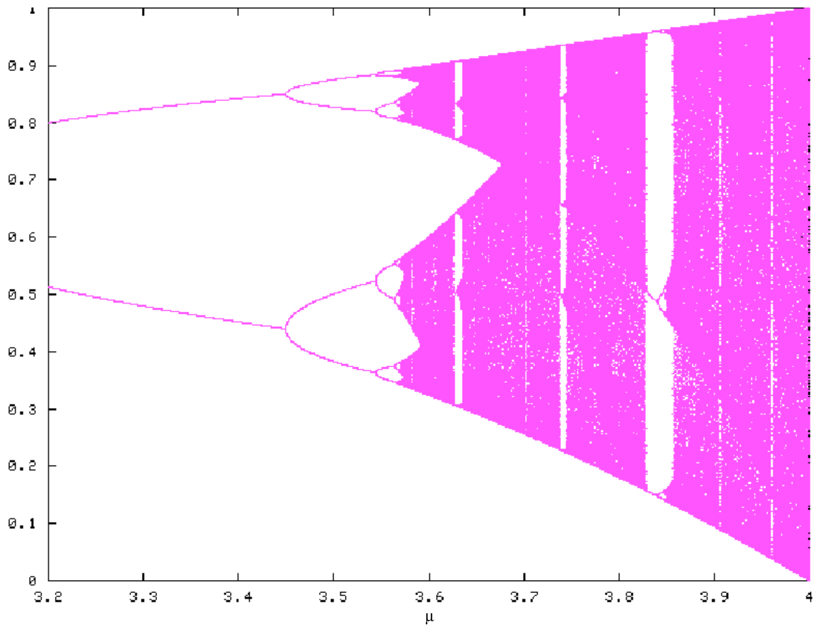


FIGURE 1.35. The bifurcation diagram of F_μ.

TABLE 1.4. Feigenbaum table.

n	μ_n	$\mu_n - \mu_{n-1}$	$\dfrac{\mu_n - \mu_{n-1}}{\mu_{n+1} - \mu_n}$
0	3	—	—
1	3.449499 ...	0.449499 ...	—
2	3.544090 ...	0.094591 ...	4.752027 ...
3	3.564407 ...	0.020313 ...	4.656673 ...
4	3.568759 ...	0.004352 ...	4.667509 ...
5	3.569692 ...	0.00093219 ...	4.668576 ...
6	3.569891 ...	0.00019964 ...	4.669354 ...

Theorem 1.25 (Feigenbaum [56] (1978)**).** *For sufficiently smooth families of maps (such as F_μ) of an interval into itself, the number $\delta = 4.6692016$ does not in general depend on the family of maps.*

1.7.4 The Bifurcation Diagram

Here the horizontal axis represents the μ values, and the vertical axis represents higher iterates $F_\mu^n(x)$. For a fixed x_0, the diagram shows the eventual behavior of $F_\mu^n(x_0)$. The bifurcation diagram was obtained with the aid of a computer for $x_0 = \frac{1}{2}$, taking increments of $\frac{1}{500}$ for $\mu \in [0, 4]$ and plotting all points $\left(\mu, F_\mu^n\left(\frac{1}{2}\right)\right)$ for $200 \le n \le 500$.

Question What happens when $\mu > \mu_\infty$?
Answer From Figure 1.35 we see that for $\mu_\infty < \mu \le 4$ we have a large number of small windows where the attracting set is an asymptotically stable cycle. The largest window appears at approximately $\mu = 3.828427\ldots$, where we have an attracting 3-cycle. Indeed, there are attracting k-cycles for all positive integers k, but their windows are so small that they may not be noticed without sufficient zooming. As in the situation where $\mu < \mu_\infty$, these k-cycles lose stability and then double bifurcate into attracting $2^n k$-cycles. We observe that outside these windows the picture looks chaotic!

Remarks: Our analysis of the logistic map F_μ may be repeated for any quadratic map $Q(x) = ax^2 + bx + c$. Indeed, the iteration of the quadratic map Q (with suitably chosen parameters) is equivalent to the iteration of the logistic map F_μ. In other words, the maps Q and F_μ possess the same type of qualitative behavior. The reader is asked, in Exercises 1.7, Problem 11, to verify that one can transform the difference equation

$$y(n+1) = y^2(n) + c \tag{1.7.7}$$

to

$$x(n+1) = \mu x(n)[1 - x(n)] \tag{1.7.8}$$

by letting

$$y(n) = -\mu x(n) + \frac{\mu}{2}, \qquad c = \frac{\mu}{2} - \frac{\mu^2}{4}. \tag{1.7.9}$$

Note here that $\mu = 2$ corresponds to $c = 0, \mu = 3$ corresponds to $c = \frac{-3}{4}$, and $\mu = 4$ corresponds to $c = -2$. Naturally, we expect to have the same behavior of the iteration of (1.7.7) and (1.7.8) at these corresponding values of μ and c.

Comments: We are still plagued by numerous unanswered questions in connection with periodic orbits (cycles) of the difference equation

$$x(n+1) = f(x(n)). \tag{1.7.10}$$

Question A. Do all points converge to some asymptotically stable periodic orbit of (1.7.8)?
The answer is definitely no.

If $f(x) = 1 - 2x^2$ in (1.7.10), then there are no asymptotically stable (attractor) periodic orbits. Can you verify this statement? If you have some difficulty here, it is not your fault. Obviously, we need some tools to help us in verifying that there are no periodic attractors.

Question B. If there is a periodic attractor of (1.7.10), how many points converge to it?
Once again, we need more machinery to answer this question.

Question C. Can there be several distinct periodic attractors for (1.7.10)?

This question leads us to the Li–Yorke famous result "Period Three Implies Chaos" [92]. To explain this and more general results requires the introduction of the so-called *Schwarzian derivative* of $f(x)$. We will come back to these questions in Chapter 6.

Exercises 1.7

Unless otherwise stated, all the problems here refer to the logistic difference equation (1.7.1).

1. Use the stair step diagram for F_4^k on $[0,1]$, $k = 1, 2, 3, \ldots$, to demonstrate that F_4 has at least 2^k periodic points of period k (including periodic points of periods that are divisors of k).

2. Find the exact solution of $x(n+1) = 4x(n)[1-x(n)]$.

3. Let $x^* = (\mu - 1)/\mu$ be the equilibrium point of (1.7.1). Show that:

   (i) For $1 < \mu \leq 3$, x^* is an attracting fixed point.

   (ii) For $\mu > 3$, x^* is a repelling fixed point.

4. Prove that $\lim_{n\to\infty} F_2^n(x) = \frac{1}{2}$ if $0 < x < 1$.

5. Let $1 < \mu \leq 2$ and let $x^* = (\mu - 1)/\mu$ be the equilibrium point of (1.7.1). Show that if $x^* < x < \frac{1}{2}$, then $\lim_{n\to\infty} F_\mu^n(x) = x^*$.

6. Prove that the 2-cycle given by (1.7.4) is attracting if $3 < \mu < 1 + \sqrt{6}$.

7. Verify formula (1.7.6). Then show that the 2-cycle in (1.7.4) is attracting when $\mu = 1 + \sqrt{6}$.

8. Verify that $\mu_2 \approx 3.54$ using a calculator or a computer.

*9. (Project). Show that the map $H_\mu(x) = \sin \mu x$ leads to the same value for the Feigenbaum number δ.

10. Show that if $|\mu - \mu_1| < \varepsilon$, then $|F\mu(x) - F\mu_1(x)| < \varepsilon$ for all $x \in [0,1]$.

11. Show that (1.7.7) can be transformed to the logistic equation (1.7.8), with $c = \frac{\mu}{2} - \frac{\mu^2}{4}$.

12. (a) Find the equilibrium points $\overset{*}{y}_1, \overset{*}{y}_2$ of (1.7.7).

    (b) Find the values of c where $\overset{*}{y}_1$ is attracting or unstable.

    (c) Find the values of c where $\overset{*}{y}_2$ is attracting or unstable.

13. Find the value of c_0 where (1.7.7) double bifurcates for $c > c_0$. Check your answer using (1.7.9).

*14. (Project). Use a calculator or a computer to develop a bifurcation diagram, as in Figures 1.34, 1.35, for (1.7.6).

*15. (Project). Develop a bifurcation diagram for the quadratic map $Q_\lambda(x) = 1 - \lambda x^2$ on the interval $[-1, 1]$, $\lambda \in (0, 2]$.

In Problems 16–19 determine the stability of the fixed points of the difference equation.

16. $x(n+1) = x(n) + \frac{1}{\pi}\sin(2\pi x(n))$.

17. $x(n+1) = 0.5\sin(\pi x(n))$.

18. $x(n+1) = 2x(n)\exp(-x(n))$.

19. A population of birds is modeled by the difference equation

$$x(n+1) = \begin{cases} 3.2x(n) & \text{for } 0 \le x(n) \le 1, \\ 0.5x(n) & \text{for } x(n) > 1, \end{cases}$$

where $x(n)$ is the number of birds in year n. Find the equilibrium points and then determine their stability.

1.8 Basin of Attraction and Global Stability (Optional)

It is customary to call an asymptotically stable fixed point or a cycle an attractor. This name makes sense since in this case all nearby points tend to the attractor. The maximal set that is attracted to an attractor M is called the *basin of attraction* of M. Our analysis applies to cycles of any period.

Definition 1.26. Let x^* be a fixed point of map f. Then the basin of attraction (or the stable set) $W^s(x^*)$ of x^* is defined as

$$W^s(x^*) = \{x : \lim_{n\to\infty} f^n(x) = x^*\}.$$

In other words, $W^s(x^*)$ consists of all points that are forward asymptotic to x^*.

Observe that if x^* is an attracting fixed point, $W^s(x^*)$ contains an open interval around x^*. The maximal interval in $W^s(x^*)$ that contains x^* is called the *immediate basin of attraction* and is denoted by $\mathcal{B}^s(x^*)$.

Example 1.27. The map $f(x) = x^2$ has one attracting fixed point $x^* = 0$. Its basin of attraction $W^s(0) = (-1, 1)$. Note that 1 is an unstable fixed point and –1 is an eventually fixed point that goes to 1 after one iteration.

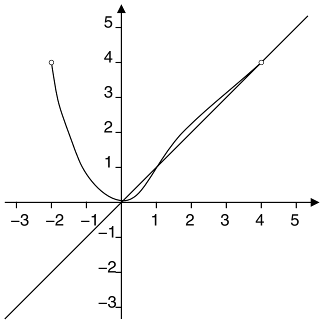


FIGURE 1.36. The basin of attraction $W^s(0) = (-1, 1)$ and $W^s(4) = [-2, -1) \cup (1, 4]$. The immediate basin of attraction $\mathcal{B}(4) = (1, 4]$.

Example 1.28. Let us now modify the map f. Consider the map $g : [-2, 4] \to [-2, 4]$ defined as

$$g(x) = \begin{cases} x^2 & \text{if } -2 \leq x \leq 1, \\ 3\sqrt{x} - 2 & \text{if } 1 < x \leq 4. \end{cases}$$

The map g has three fixed points $x_1^* = 0$, $x_2^* = 1$, $x_3^* = 4$. The basin of attraction of $x_1^* = 0$, $W^s(0) = (-1, 1)$, while the basin of attraction of $x_3^* = 4$, $W^s(4) = [-2, -1) \cup (1, 4]$. Moreover, the immediate basin of attractions of $x_1^* = 0$ is $\mathcal{B}(0) = W^s(0) = (-1, 1)$, while $\mathcal{B}(4) = (1, 4]$.

Remark: Observe that in the preceding example, the basins of attraction of the two fixed points $x_1^* = 0$ and $x_3^* = 4$ are disjoint. This is no accident and is, in fact, generally true. This is due to the uniqueness of a limit of a sequence. In other words, if the $\lim_{n\to\infty} f^n(x) = L_1$ and $\lim_{n\to\infty} f^n(x) = L_2$, then certainly $L_1 = L_2$.

It is worth noting here that finding the basin of attraction of a fixed point is in general a difficult task. But even more difficult is providing a rigorous proof. The most efficient method to determining the basin of attraction is the method of Liapunov functions, which will be developed in Chapter 4. In this section, we will develop some of the basic topological properties of the basin of attractions. Henceforth, all our maps are assumed to be *continuous*. We begin our exposition by defining the important notion of invariance.

Definition 1.29. A set M is *positively invariant* under a map f if $f(M) \subseteq M$. In other words, for every $x \in M$, $\mathcal{O}(x) \subseteq M$. Since we are only considering forward iterations of f, the prefix "positively" will henceforth be dropped.

Clearly an orbit of a point is invariant.

Next we show that the basin of attraction of an attracting fixed point is invariant and open.

Theorem 1.30. *Let $f : I \to I$, $I = [a, b]$, be a continuous map and let $x^* \in [a, b]$ be a fixed point of f. Then the following statements hold true:*

(i) *The immediate basin of attraction $\mathcal{B}(x^*)$ is an interval containing x^*, which is either an open interval (c, d) or of the form $[a, c)(c, b]$. Moreover, $\mathcal{B}(x^*)$ is invariant.*

(ii) *$W^s(x^*)$ is invariant. Furthermore, $W^s(x^*)$ is the union (maybe an infinite union) of intervals that are either open intervals or of the form $[a, c)$ or $(d, b]$.*

PROOF.

(i) We know that $\mathcal{B}(x^*)$ is a maximal interval in $W^s(x^*)$ containing x^*. Assume that $\mathcal{B}(x^*) = [c, d)$, $c \neq a$. Now for a given small $\varepsilon > 0$ there exists $m \in \mathbb{Z}^+$ such that $f^m(c) \in (x^* - \varepsilon, x^* + \varepsilon) \subset (c, d)$. Since f^m is continuous, there exists $\delta > 0$ such that if $x_0 \in (c - \delta, c + \delta)$, then $f^m(x_0) \in (x^* - \varepsilon, x^* + \varepsilon) \subset \mathcal{B}(x^*)$. Then $x_0 \in \mathcal{B}(x^*)$ and hence $(c - \delta, d) \subset W^s(x^*)$ which violates the maximality of $\mathcal{B}(x^*)$. Hence $\mathcal{B}(x^*) \neq [c, a)$, a contradiction. Analogously, one may show that $W^s(x^*) \neq (c, d]$ if $d \neq b$.

To prove the invariance of $\mathcal{B}(x^*)$, assume that there exists $y \in \mathcal{B}(x^*)$ such that $f^r(y) \notin \mathcal{B}(x^*)$ for some $r \in \mathbb{Z}^+$. Since $\mathcal{B}(x^*)$ is an interval, it follows by the Intermediate Value Theorem that $f^r(\mathcal{B}(x^*))$ is also an interval. Moreover, this interval $f^r(\mathcal{B}(x^*))$ must contain x^* since $f^r(x^*) = x^*$. Thus $f^r(\mathcal{B}(x^*)) \cap \mathcal{B}(x^*) \neq 0$, and hence $\mathcal{B}(x^*) \cup f^r(\mathcal{B}(x^*))$ is an interval in $W^s(x^*)$, which violates the maximality of $\mathcal{B}(x^*)$.

(ii) The proof of this part is analogous to the proof of part (a) and will be left to the reader to verify. □

There are several (popular) maps such as the logistic map and Ricker's map in which the basin of attraction, for the attractive fixed point, is the entire space with the exception of one or two points (fixed or eventually fixed). For the logistic map $F_\mu(x) = \mu x(1 - x)$ and $1 < \mu < 3$, the basin of attraction $W^s(x^*) = (0, 1)$ for the fixed point $x^* = \frac{\mu-1}{\mu}$. And for Ricker's map $R_p(x) = xe^{p-x}$, $0 < p < 2$, the basin of attraction $W^s(x^*) = (0, \infty)$, for $x^* = p$. Here we will consider only the logistic map and leave it to the reader to prove the statement concerning Ricker's map.

Notice that $|F'_\mu(x)| = |\mu - 2\mu x| < 1$ if and only if $-1 < \mu - 2\mu x < 1$. This implies that $\frac{\mu-1}{2\mu} < x < \frac{\mu+1}{2\mu}$. Hence $|F'_\mu(x)| < 1$ for all $x \in \left(\frac{\mu-1}{2\mu}, \frac{\mu+1}{2\mu}\right)$. Observe that $x^* = \frac{\mu-1}{\mu} \in \left(\frac{\mu-1}{2\mu}, \frac{\mu+1}{2\mu}\right)$ if and only if $1 < \mu < 3$. Now

$F_\mu\left(\frac{\mu+1}{2\mu}\right) = F_\mu\left(\frac{\mu-1}{2\mu}\right) = \frac{1}{2}\left[\frac{(\mu-1)(\mu+1)}{2\mu}\right]$. Notice that since $1 < \mu < 3$, $\frac{\mu-1}{2\mu} < \frac{1}{2} \cdot \frac{(\mu-1)(\mu+1)}{2\mu} < \frac{\mu+1}{2\mu}$. Hence $\left[\frac{\mu-1}{2\mu}, \frac{\mu+1}{2\mu}\right] \subset W^s(x^*)$.

If $z \in \left(0, \frac{\mu-1}{2\mu}\right)$, then $F'_\mu(z) > 1$. By the Mean Value Theorem, $\frac{F_\mu(z)-F_\mu(0)}{z-0} = F'_\mu(\gamma)$, for some γ with $0 < \gamma < z$. Hence

$$F_\mu(z) - F_\mu(0) = F_\mu(z) \geq \beta z$$

for some $\beta > 1$. Then for some $r \in \mathbb{Z}^+$, $F^r_\mu(z) \geq \beta^r z > \frac{\mu-1}{2\mu}$ and $F^{r-1}_\mu(z) < \frac{\mu-1}{2\mu}$. Moreover, since F is increasing on $\left[0, \frac{\mu-1}{2\mu}\right]$, $F^r_\mu(z) < F_\mu\left(\frac{\mu-1}{2\mu}\right) = \mu\left(\frac{\mu-1}{2\mu}\right)\left(1 - \frac{\mu-1}{2\mu}\right) = \frac{\mu-1}{\mu}\left(\frac{\mu+1}{4}\right) \leq x^*$. Thus $z \in W^s(x^*)$. On the other hand, $F_\mu\left(\frac{\mu+1}{2\mu}, 1\right) \subset (0, x^*)$ and hence $\left(\frac{\mu+1}{2\mu}, 1\right) \subset W^s(x^*)$. This shows that $W^s(x^*) = (0,1)$.

To summarize

Lemma 1.31. *For the logistic map $F_\mu(x) = \mu x(1-x)$, $1 < \mu < 3$, $W^s(x^*) = (0,1)$ for $x^* = \frac{\mu-1}{\mu}$.*

We now turn our attention to periodic points. If $\bar{x}$ is a periodic point of period k under the map f, then its basin of attraction $W^s(\bar{x})$ is its basin of attraction as a fixed point under the map f^k. Hence $W^s(\bar{x}) = \{x : \lim_{n\to\infty} (f^k)^n(x) = \lim_{n\to\infty} f^{kn}(x) = \bar{x}\}$. Let $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k\}$ be a k-cycle of a map f. Then clearly for $i \neq j$, $W^s(\bar{x}_i) \cap W^s(\bar{x}_j) = \emptyset$. (Why?) More generally, if x is a periodic point of period r and $y \neq x$ is a periodic point of period s, then $W^s(x) \cap W^s(y) = \emptyset$ (Exercises 1.8, Problem 6).

Example 1.32. Consider the function $f(x) = -x^{\frac{1}{3}}$. Then $x^* = 0$ is the only fixed point. There is a 2-cycle $\{-1, 1\}$ with $f(-1) = 1$, $f^2(-1) = -1$. The cobweb diagram (Figure 1.37) shows that $W^s(1) = (0, \infty)$, $W^s(-1) = (-\infty, 0)$.

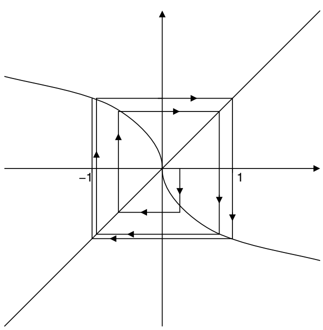


FIGURE 1.37.

Exercises 1.8

1. Investigate the basin of attraction of the fixed points of the map

$$f(x) = \begin{cases} x^2 & \text{if } -3 \le x \le 1, \\ 4\sqrt{x} - 3 & \text{if } 1 < x \le 9. \end{cases}$$

2. Let $f(x) = |x - 1|$. Find $W^s(\frac{1}{2})$.

3. Suppose that $f : I \to I$ is a continuous and onto map on an interval I. Let $\bar{x}$ be an asymptotically stable periodic point of period $k \ge 2$. Show that $W^s(f(\bar{x})) = f(W^s(\bar{x}))$.

4. Describe the basin of attraction of all fixed and periodic points of the maps:

   (i) $f(x) = x^2$,

   (ii) $g(x) = x^3$,

   (iii) $h(x) = 2xe^{-x}$,

   (iv) $q(x) = -\frac{4}{\pi} \arctan x$.

5. Investigate the basin of attraction of the origin for the map

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{if } 0 \le x \le 0.2, \\ 3x - \dfrac{1}{2} & \text{if } 0.2 < x \le \dfrac{1}{2}, \\ 2 - 2x & \text{if } \dfrac{1}{2} < x \le 1. \end{cases}$$

6. Let f be a continuous map that has two periodic points x and y, $x \ne y$, with periods r and t, $r \ne t$, respectively. Prove that $W^s(x) \cap W^s(y) = \emptyset$.

7*. Suppose that a set M is invariant under a one-to-one continuous map f. A point $x \in M$ is said to be an interior point if $(x - \delta, x + \delta) \subset M$ for some $\delta > 0$. Prove that the set of all interior points of M, denoted by $\text{int}(M)$, is invariant.

8. Let x^* be an attracting fixed point under a continuous map f. If the immediate basin of attraction $\mathcal{B}(x^*) = (a, b)$, show that the set $\{a, b\}$ is invariant. Then conclude that there are only three scenarios in this case: (1) both a and b are fixed points, or (2) a or b is fixed and the other is an eventually fixed point, or (3) $\{a, b\}$ is a 2-cycle.

9. Show that for Ricker's map

$$R_p(x) = xe^{p-x}, \quad 0 < p < 2,$$
$$W^s(x^*) = (0, \infty), \quad \text{where } x^* = p.$$

10. (Term Project). Consider the logistic map $F_\mu(x) = \mu x(1 - x)$ with $3 < \mu < 1 + \sqrt{6}$. Let $c = \{\bar{x}_1, \bar{x}_2\}$ be the attracting 2-cycle. Show that

$W^s(c) = W^s(\bar{x}_1) \cup W^s(\bar{x}_2)$ is all the points in $(0, 1)$ except the set of eventually fixed points (including the fixed point $\frac{\mu-1}{\mu}$).

2

Linear Difference Equations of Higher Order

In this chapter we examine linear difference equations of high order, namely, those involving a single dependent variable.[1] Such equations arise in almost every field of scientific inquiry, from population dynamics (the study of a single species) to economics (the study of a single commodity) to physics (the study of the motion of a single body). We will become acquainted with some of these applications in this chapter. We start this chapter by introducing some rudiments of difference calculus that are essential in the study of linear equations.

2.1 Difference Calculus

Difference calculus is the discrete analogue of the familiar differential and integral calculus. In this section we introduce some very basic properties of two operators that are essential in the study of difference equations. These are the *difference operator* (Section 1.2)

$$\Delta x(n) = x(n+1) - x(n)$$

and the *shift operator*

$$Ex(n) = x(n+1).$$

[1] Difference equations that involve more than one dependent variable are called systems of difference equations; we will inspect these equations in Chapter 3.

It is easy to see that

$$E^k x(n) = x(n+k).$$

However, $\Delta^k x(n)$ is not so apparent. Let I be the *identity operator*, i.e., $Ix = x$. Then, one may write $\Delta = E - I$ and $E = \Delta + I$.

Hence,

$$\begin{aligned}\Delta^k x(n) &= (E-I)^k x(n)\\ &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} E^{k-i} x(n),\end{aligned}$$

$$\boxed{\Delta^k x(n) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} x(n+k-i).} \tag{2.1.1}$$

Similarly, one may show that

$$E^k x(n) = \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} x(n). \tag{2.1.2}$$

We should point out here that the operator Δ is the counterpart of the derivative operator D in calculus. Both operators E and Δ share one of the helpful features of the derivative operator D, namely, the property of *linearity.*

"Linearity" simply means that $\Delta[ax(n)+by(n)] = a\Delta x(n) + b\Delta y(n)$ and $E[ax(n)+by(n)] = aEx(n) + bEy(n)$, for all a and $b \in \mathbb{R}$. In Exercises 2.1, Problem 1, the reader is allowed to show that both Δ and E are linear operators.

Another interesting difference, parallel to differential calculus, is the discrete analogue of the fundamental theorem of calculus. [2]

Lemma 2.1. *The following statements hold:*

(i)

$$\boxed{\sum_{k=n_0}^{n-1} \Delta x(k) = x(n) - x(n_0),} \tag{2.1.3}$$

---

[2]The fundamental theorem of calculus states that:

(i) $\int_a^b df(x) = f(b) - f(a)$,

(ii) $d\left(\int_a^x f(t)\,dt\right) = f(x)$.

(ii)

$$\boxed{\Delta\left(\sum_{k=n_0}^{n-1} x(k)\right) = x(n).} \tag{2.1.4}$$

PROOF. The proof remains as Exercises 2.1, Problem 3. □

We would now like to introduce a third property that the operator Δ has in common with the derivative operator D.

Let

$$p(n) = a_0 n^k + a_1 n^{k-1} + \cdots + a_k$$

be a polynomial of degree k. Then

$$\begin{aligned}\Delta p(n) &= \left[a_0(n+1)^k + a_1(n+1)^{k-1} + \cdots + a_k\right] \\ &\quad - \left[a_0 n^k + a_1 n^{k-1} + \cdots + a_k\right] \\ &= a_0 k n^{k-1} + \text{ terms of degree lower than } (k-1).\end{aligned}$$

Similarly, one may show that

$$\Delta^2 p(n) = a_0 k(k-1) n^{k-2} + \text{ terms of degree lower than } (k-2).$$

Carrying out this process k times, one obtains

$$\Delta^k p(n) = a_0 k!. \tag{2.1.5}$$

Thus,

$$\Delta^{k+i} p(n) = 0 \text{ for } i \geq 1. \tag{2.1.6}$$

2.1.1 The Power Shift

We now discuss the action of a polynomial of degree k in the shift operator E on the term b^n, for any constant b.

Let

$$p(E) = a_0 E^k + a_1 E^{k-1} + \cdots + a_k I \tag{2.1.7}$$

be a polynomial of degree k in E.

Then

$$\begin{aligned}p(E)b^n &= a_0 b^{n+k} + a_1 b^{n+k-1} + \cdots + a_k b^n \\ &= (a_0 b^k + a_1 b^{k-1} + \cdots + a_k) b^n \\ &= p(b) b^n.\end{aligned} \tag{2.1.8}$$

A generalization of formula (2.1.8) now follows.

Lemma 2.2. *Let $p(E)$ be the polynomial in* (2.1.7) *and let $g(n)$ be any discrete function. Then*

$$\boxed{p(E)(b^n g(n)) = b^n p(bE)g(n).} \tag{2.1.9}$$

Proof. This is left to the reader as Exercises 2.1, Problem 4. □

2.1.2 Factorial Polynomials

One of the most interesting functions in difference calculus is the *factorial polynomial* $x^{(k)}$ defined as follows. Let $x \in \mathbb{R}$. Then the kth factorial of x is given by

$$x^{(k)} = x(x-1)\cdots(x-k+1), \qquad k \in \mathbb{Z}^+.$$

Thus if $x = n \in \mathbb{Z}^+$ and $n \geq k$, then

$$\boxed{n^{(k)} = \frac{n!}{(n-k)!}}$$

and

$$n^{(n)} = n!.$$

The function $x^{(k)}$ plays the same role here as that played by the polynomial x^k in differential calculus. The following Lemma 2.3 demonstrates this fact.

So far, we have defined the operators Δ and E on sequences $f(n)$. One may extend the definitions of Δ and E to continuous functions $f(t)$, $t \in \mathbb{R}$, by simply letting $\Delta f(t) = f(t+1) - f(t)$ and $Ef(t) = f(t+1)$. This extension enables us to define $\Delta f(x)$ and $Ef(x)$ where $f(x) = x^{(k)}$ by

$$\Delta x^{(k)} = (x+1)^{(k)} - x^{(k)} \quad \text{and} \quad Ex^{(k)} = (x+1)^{(k)}.$$

Using this definition one may establish the following result.

Lemma 2.3. *For fixed $k \in \mathbb{Z}^+$ and $x \in \mathbb{R}$, the following statements hold:*

(i)

$$\boxed{\Delta x^{(k)} = kx^{(k-1)};} \tag{2.1.10}$$

(ii)

$$\boxed{\Delta^n x^{(k)} = k(k-1),\ldots,(k-n+1)x^{(k-n)};} \tag{2.1.11}$$

(iii)

$$\boxed{\Delta^k x^{(k)} = k!.} \tag{2.1.12}$$

Proof. (i)

$$\begin{aligned}\Delta x^{(k)} &= (x+1)^{(k)} - x^{(k)}\\ &= (x+1)x(x-1)\cdots(x-k+2) - x(x-1)\\ &\quad\cdots(x-k+2)(x-k+1)\\ &= x(x-1)\cdots(x-k+2)\cdot k\\ &= kx^{(k-1)}.\end{aligned}$$

The proofs of parts (ii) and (iii) are left to the reader as Exercises 2.1, Problem 5. □

If we define, for $k \in \mathbb{Z}^+$,

$$x^{(-k)} = \frac{1}{x(x+1)\cdots(x+k-1)} \tag{2.1.13}$$

and $x^{(0)} = 1$, then one may extend Lemma 2.3 to hold for all $k \in \mathbb{Z}$. In other words, parts (i), (ii), and (iii) of Lemma 2.3 hold for all $k \in \mathbb{Z}$ (Exercises 2.1, Problem 6).

The reader may wonder whether the product and quotient rules of the differential calculus have discrete counterparts. The answer is affirmative, as may be shown by the following two formulas, where proofs are left to the reader as Exercises 2.1, Problem 7.

Product Rule:

$$\boxed{\Delta[x(n)y(n)] = Ex(n)\Delta y(n) + y(n)\Delta x(n).} \tag{2.1.14}$$

Quotient Rule:

$$\boxed{\Delta\left[\frac{x(n)}{y(n)}\right] = \frac{y(n)\Delta x(n) - x(n)\Delta y(n)}{y(n)Ey(n)}.} \tag{2.1.15}$$

2.1.3 The Antidifference Operator

The discrete analogue of the indefinite integral in calculus is the antidifference operator Δ^{-1}, defined as follows. If $\Delta F(n) = 0$, then $\Delta^{-1}(0) = F(n) = c$ for some arbitrary constant c. Moreover, if $\Delta F(n) = f(n)$, then $\Delta^{-1} f(n) = F(n) + c$, for some arbitrary constant c. Hence

$$\begin{aligned}\Delta\Delta^{-1} f(n) &= f(n),\\ \Delta^{-1}\Delta F(n) &= F(n) + c,\end{aligned}$$

and

$$\Delta\Delta^{-1} = I \quad \text{but} \quad \Delta^{-1}\Delta \neq I.$$

Using formula (2.1.4) one may readily obtain

$$\boxed{\Delta^{-1} f(n) = \sum_{i=0}^{n-1} f(i) + c.} \tag{2.1.16}$$

Formula (2.1.16) is very useful in proving that the operator Δ^{-1} is linear.

Theorem 2.4. *The operator Δ^{-1} is linear.*

PROOF. We need to show that for $a, b \in \mathbb{R}$, $\Delta^{-1}[ax(n) + by(n)] = a\Delta^{-1}x(n) + b\Delta^{-1}y(n)$. Now, from formula (2.1.16) we have

$$\begin{aligned} \Delta^{-1}[ax(n) + by(n)] = & \sum_{i=0}^{n-1} ax(i) + by(i) + c \\ &= a\sum_{i=0}^{n-1} x(i) + b\sum_{i=0}^{n-1} y(i) + c \\ &= a\Delta^{-1}x(n) + b\Delta^{-1}y(n). \end{aligned}$$

□

Next we derive the antidifference of some basic functions.

Lemma 2.5. *The following statements hold:*

(i)

$$\Delta^{-k} 0 = c_1 n^{k-1} + c_2 n^{k-2} + \cdots + c_k. \tag{2.1.17}$$

(ii)

$$\Delta^{-k} 1 = \frac{n^k}{k!} + c_1 n^{k-1} + c_2 n^{k-2} + \cdots + c_k. \tag{2.1.18}$$

(iii)

$$\boxed{\Delta^{-1} n^{(k)} = \frac{n^{(k+1)}}{k+1} + c, \qquad k \neq -1.} \tag{2.1.19}$$

PROOF. The proofs of parts (i) and (ii) follow by applying Δ^k to the right-hand side of formulas (2.1.17) and (2.1.18) and then applying formulas (2.1.6) and (2.1.5), respectively. The proof of part (iii) follows from formula (2.1.10).

Finally, we give the discrete analogue of the integration by parts formula, namely, the summation by parts formula:

$$\boxed{\sum_{k=0}^{n-1} y(k)\Delta x(k) = x(n)y(n) - \sum_{k=0}^{n-1} x(k+1)\Delta y(k) + c.} \tag{2.1.20}$$

To prove formula (2.1.20) we use formula (2.1.14) to obtain

$$y(n)\Delta x(n) = \Delta(x(n)y(n)) - x(n+1)\Delta y(n).$$

Applying Δ^{-1} to both sides and using formula (2.1.16), we get

$$\sum_{k=0}^{n-1} y(k)\Delta x(k) = x(n)y(n) - \sum_{k=0}^{n-1} x(k+1)\Delta y(k) + c. \qquad \square$$

Exercises 2.1

1. Show that the operators Δ and E are linear.
2. Show that $E^k x(n) = \sum_{i=0}^{k} \binom{k}{i} \Delta^{k-i} x(n)$.
3. Verify formulas (2.1.3) and (2.1.4).
4. Verify formula (2.1.9).
5. Verify formulas (2.1.11) and (2.1.12).
6. Show that Lemma 2.3 holds for $k \in \mathbb{Z}$.
7. Verify the product and quotient rules (2.1.14) and (2.1.15).
8. (Abel's Summation Formula). Prove that

$$\boxed{\sum_{k=1}^{n} x(k)y(k) = x(n+1)\sum_{k=1}^{n} y(k) - \sum_{k=1}^{n}\left(\Delta x(k)\sum_{r=1}^{k} y(r)\right).}$$

9. (Newton's Theorem). If $f(n)$ is a polynomial of degree k, show that

$$f(n) = f(0) + \frac{n^{(1)}}{1!}\Delta f(0) + \frac{n^{(2)}}{2!}\Delta^2 f(0) + \cdots + \frac{n^{(k)}}{k!}\Delta^{(k)} f(0).$$

10. (The Discrete Taylor Formula). Verify that

$$\boxed{f(n) = \sum_{i=0}^{k-1} \binom{n}{i} \Delta^i f(0) + \sum_{s=0}^{n-k} \binom{n-s-1}{k-1} \Delta^k f(s).}$$

11. (The Stirling Numbers). The Stirling numbers of the second kind $s_i(k)$ are defined by the difference equation $s_i(m+1) = s_{i-1}(m) + i s_i(m)$ with $s_i(i) = s_1(i) = 1$ and $1 \le i \le m$, $s_1(k) = 0$ for $1 > k$. Prove that

$$x^m = \sum_{i=1}^{m} s_i(m) x^{(i)}. \tag{2.1.21}$$

12. Use (2.1.21) to verify Table 2.1 which gives the Stirling numbers $s_i(k)$ for $1 \le i, k \le 7$.
13. Use Table 2.1 and formula (2.1.21) to write $x^3, x^4,$ and x^5 in terms of the factorial polynomials $x^{(k)}$ (e.g., $x^2 = x^{(1)} + x^{(2)}$).
14. Use Problem 13 to find

TABLE 2.1. Stirling numbers $s_i(k)$.

$i\backslash k$	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2		1	3	7	15	31	63
3			1	6	25	90	301
4				1	10	65	350
5					1	15	140
6						1	21
7							1

(i) $\Delta^{-1}(n^3+1)$.

(ii) $\Delta^{-1}\left(\dfrac{5}{n(n+3)}\right)$.

15. Use Problem 13 to solve the difference equation $y(n+1) = y(n) + n^3$.
16. Use Problem 13 to solve the difference equation $y(n+1) = y(n) - 5n^2$.
17. Consider the difference equation[3]

$$y(n+1) = a(n)y(n) + g(n). \tag{2.1.22}$$

(a) Put $y(n) = \left(\prod_{i=0}^{n-1} a(i)\right) u(n)$ in (2.1.22). Then show that $\Delta u(n) = g(n)/\prod_{i=0}^{n} a(i)$.

(b) Prove that

$$y(n) = \left(\prod_{i=0}^{n-1} a(i)\right) y_0 + \sum_{r=0}^{n-1}\left(\prod_{i=r+1}^{n-1} a(i)\right) g(r), \qquad y_0 = y(0).$$

(Compare with Section 1.2.)

2.2 General Theory of Linear Difference Equations

The normal form of a kth-order *nonhomogeneous linear* difference equation is given by

$$y(n+k) + p_1(n)\,y(n+k-1) + \cdots + p_k(n)\,y(n) = g(n), \tag{2.2.1}$$

where $p_i(n)$ and $g(n)$ are real-valued functions defined for $n \geq n_0$ and $p_k(n) \neq 0$ for all $n \geq n_0$. If $g(n)$ is identically zero, then (2.2.1) is said to be a homogeneous equation. Equation (2.2.1) may be written in the form

$$y(n+k) = -p_1(n)\,y(n+k-1) - \cdots - p_k(n)\,y(n) + g(n). \tag{2.2.2}$$

[3]This method of solving a nonhomogeneous equation is called the method of variation of constants.

By letting $n = 0$ in (2.2.2), we obtain $y(k)$ in terms of $y(k-1), y(k-2), \cdots, y(0)$. Explicitly, we have

$$y(k) = -p_1(0)y(k-1) - p_2(0)y(k-2) - \cdots - p_k(0)y(0) + g(0).$$

Once $y(k)$ is computed, we can go to the next step and evaluate $y(k+1)$ by letting $n = 1$ in (2.2.2). This yields

$$y(k+1) = -p_1(1)y(k) - p_2(1)y(k-1) - \cdots - p_k(1)y(1) + g(1).$$

By repeating the above process, it is possible to evaluate all $y(n)$ for $n \geq k$. Let us now illustrate the above procedure by an example.

Example 2.6. Consider the third-order difference equation

$$y(n+3) - \frac{n}{n+1}y(n+2) + ny(n+1) - 3y(n) = n, \tag{2.2.3}$$

where $y(1) = 0, y(2) = -1$, and $y(3) = 1$. Find the values of $y(4)$, $y(5)$, $y(6)$, and $y(7)$.

Solution First we rewrite (2.2.3) in the convenient form

$$y(n+3) = \frac{n}{n+1}y(n+2) - ny(n+1) + 3y(n) + n. \tag{2.2.4}$$

Letting $n = 1$ in (2.2.4), we have

$$y(4) = \frac{1}{2}y(3) - y(2) + 3y(1) + 1 = \frac{5}{2}.$$

For $n = 2$,

$$y(5) = \frac{2}{3}y(4) - 2y(3) + 3y(2) + 2 = -\frac{4}{3}.$$

For $n = 3$,

$$y(6) = \frac{3}{4}y(5) - 3y(4) + 3y(3) + 3 = -\frac{3}{2}.$$

For $n = 4$,

$$y(7) = \frac{4}{5}y(6) - 4y(5) + 3y(4) + 4 = 20.9.$$

Now let us go back to (2.2.1) and formally define its solution. A sequence $\{y(n)\}_{n_0}^{\infty}$ or simply $y(n)$ is said to be a *solution* of (2.2.1) if it satisfies the equation. Observe that if we specify the initial data of the equation, we are led to the corresponding initial value problem

$$y(k+n) + p_1(n)y(n+k-1) + \cdots + p_k(n)y(n) = g(n), \tag{2.2.5}$$

$$y(n_0) = a_0, \quad y(n_0+1) = a_1, \ldots, y(n_0+k-1) = a_{k-1}, \tag{2.2.6}$$

where the a_i's are real numbers. In view of the above discussion, we conclude with the following result.

Theorem 2.7. *The initial value problems* (2.2.5) *and* (2.2.6) *have a* unique *solution* $y(n)$.

PROOF. The proof follows by using (2.2.5) for $n = n_0, n_0 + 1, n_0 + 2, \ldots$. Notice that any $n \geq n_0 + k$ may be written in the form $n = n_0 + k + (n - n_0 - k)$. By *uniqueness* of the solution $y(n)$ we mean that if there is another solution $\tilde{y}(n)$ of the initial value problems (2.2.5) and (2.2.6), then $\tilde{y}(n)$ must be identical to $y(n)$. This is again easy to see from (2.2.5). □

The question still remains whether we can find a closed-form solution for (2.2.1) or (2.2.5) and (2.2.6). Unlike our amiable first-order equations, obtaining a closed-form solution of (2.2.1) is a formidable task. However, if the coefficients p_i in (2.2.1) are constants, then a solution of the equation may be easily obtained, as we see in the next section.

In this section we are going to develop the general theory of kth-order linear *homogeneous* difference equations of the form

$$x(n+k) + p_1(n)x(n+k-1) + \cdots + p_k(n)x(n) = 0. \qquad (2.2.7)$$

We start our exposition by introducing three important definitions.

Definition 2.8. The functions $f_1(n), f_2(n), \ldots, f_r(n)$ are said to be *linearly dependent* for $n \geq n_0$ if there are constants $a_1, a_2, \ldots, a_r$, not all zero, such that

$$a_1 f_1(n) + a_2 f_2(n) + \cdots + a_r f_r(n) = 0, \qquad n \geq n_0.$$

If $a_j \neq 0$, then we may divide (2.2.7) by a_j to obtain

$$\begin{aligned} f_j(n) &= -\frac{a_1}{a_j} f_1(n) - \frac{a_2}{a_j} f_2(n) \cdots - \frac{a_r}{a_j} f_r(n) \\ &= -\sum_{i \neq j} \frac{a_i}{a_j} f_i(n). \end{aligned} \qquad (2.2.8)$$

Equation (2.2.8) simply says that each f_j with nonzero coefficient is a *linear combination* of the other f_i's. Thus two functions $f_1(n)$ and $f_2(n)$ are linearly dependent if one is a multiple of the other, i.e., $f_1(n) = af_2(n)$, for some constant a.

The negation of linear dependence is *linear independence.* Explicitly put, the functions $f_1(n), f_2(n), \ldots, f_r(n)$ are said to be *linearly independent* for $n \geq n_0$ if whenever

$$a_1 f_1(n) + a_2 f_2(n) + \cdots + a_r f_r(n) = 0$$

for all $n \geq n_0$, then we must have $a_1 = a_2 = \cdots = a_r = 0$.

Let us illustrate this new concept by an example.

Example 2.9. Show that the functions 3^n, $n3^n$, and $n^2 3^n$ are linearly independent on $n \geq 1$.

Solution Suppose that for constants a_1, a_2, and a_3 we have

$$a_1 3^n + a_2 n 3^n + a_3 n^2 3^n = 0, \qquad \text{for all } n \geq 1.$$

Then by dividing by 3^n we get

$$a_1 + a_2 n + a_3 n^2 = 0, \qquad \text{for all } n \geq 1.$$

This is impossible unless $a_3 = 0$, since a second-degree equation in n possesses at most two solutions $n \geq 1$. Hence $a_1 = a_2 = a_3 = 0$. Similarly, $a_2 = 0$, whence $a_1 = 0$, which establishes the linear independence of our functions.

Definition 2.10. A set of k linearly independent solutions of (2.2.7) is called a *fundamental set* of solutions.

As you may have noticed from Example 2.9, it is not practical to check the linear independence of a set of solutions using the definition. Fortunately, there is a simple method to check the linear independence of solutions using the so-called Casoratian $W(n)$, which we now define for the eager reader.

Definition 2.11. The Casoratian[4] $W(n)$ of the solutions $x_1(n), x_2(n), \ldots, x_r(n)$ is given by

$$W(n) = \det \begin{pmatrix} x_1(n) & x_2(n) & \ldots & x_r(n) \\ x_1(n+1) & x_2(n+1) & \ldots & x_r(n+1) \\ \vdots & & & \\ x_1(n+r-1) & x_2(n+r-1) & \ldots & x_r(n+r-1) \end{pmatrix}. \tag{2.2.9}$$

Example 2.12. Consider the difference equation

$$x(n+3) - 7x(n+1) + 6x(n) = 0.$$

(a) Show that the sequences 1, $(-3)^n$, and 2^n are solutions of the equation.

(b) Find the Casoratian of the sequences in part (a).

Solution

(a) Note that $x(n) = 1$ is a solution, since $1 - 7 + 6 = 0$. Furthermore, $x(n) = (-3)^n$ is a solution, since

$$(-3)^{n+3} - 7(-3)^{n+1} + 6(-3)^n = (-3)^n[-27 + 21 + 6] = 0.$$

Finally, $x(n) = 2^n$ is a solution, since

$$(2)^{n+3} - 7(2)^{n+1} + 6(2)^n = 2^n[8 - 14 + 6] = 0.$$

[4]This is the discrete analogue of the Wronskian in differential equations.

(b) Now,

$$
\begin{aligned}
W(n) &= \det \begin{pmatrix} 1 & (-3)^n & 2^n \\ 1 & (-3)^{n+1} & 2^{n+1} \\ 1 & (-3)^{n+2} & 2^{n+2} \end{pmatrix} \\
&= \begin{vmatrix} (-3)^{n+1} & (2)^{n+1} \\ (-3)^{n+2} & (2)^{n+2} \end{vmatrix} - (-3)^n \begin{vmatrix} 1 & (2)^{n+1} \\ 1 & (2)^{n+2} \end{vmatrix} \\
&\quad + (2)^n \begin{vmatrix} 1 & (-3)^{n+1} \\ 1 & (-3)^{n+2} \end{vmatrix} \\
&= (2)^{n+2}(-3)^{n+1} - (2)^{n+1}(-3)^{n+2} - (-3)^n((2)^{n+2} - (2)^{n+1}) \\
&\quad + (2)^n((-3)^{n+2} - (-3)^{n+1}) \\
&= -12(2)^n(-3)^n - 18(2)^n(-3)^n - 4(2)^n(-3)^n \\
&\quad + 2(2)^n(-3)^n + 9(2)^n(-3)^n + 3(2)^n(-3)^n \\
&= -20(2)^n(-3)^n.
\end{aligned}
$$

Next we give a formula, called Abel's formula, to compute the Casoratian $W(n)$. The significance of Abel's formula is its effectiveness in the verification of the linear independence of solutions.

Lemma 2.13 (Abel's Lemma). *Let $x_1(n), x_2(n), \ldots, x_k(n)$ be solutions of* (2.2.7) *and let $W(n)$ be their Casoratian. Then, for $n \geq n_0$,*

$$
\boxed{W(n) = (-1)^{k(n-n_0)} \left(\prod_{i=n_0}^{n-1} p_k(i) \right) W(n_0).} \tag{2.2.10}
$$

Proof. We will prove the lemma for $k = 3$, since the general case may be established in a similar fashion. So let $x_1(n)$, $x_2(n)$, and $x_3(n)$ be three independent solutions of (2.2.7). Then from formula (2.2.9) we have

$$
W(n+1) = \det \begin{pmatrix} x_1(n+1) & x_2(n+1) & x_3(n+1) \\ x_1(n+2) & x_2(n+2) & x_3(n+2) \\ x_1(n+3) & x_2(n+3) & x_3(n+3) \end{pmatrix}. \tag{2.2.11}
$$

From (2.2.7) we have, for $1 \leq i \leq 3$,

$$
x_i(n+3) = -p_3(n)x_i(n) - [p_1(n)x_i(n+2) + p_2(n)x_i(n+1)]. \tag{2.2.12}
$$

Now, if we use formula (2.2.12) to substitute for $x_1(n+3)$, $x_2(n+3)$, and $x_3(n+3)$ in the last row of formula (2.2.11), we obtain

$$W(n+1) = \det \begin{pmatrix} x_1(n+1) & x_2(n+1) & x_3(n+1) \\ x_1(n+2) & x_2(n+2) & x_3(n+2) \\ -p_3 x_1(n) & -p_3 x_2(n) & -p_3 x_3(n) \\ -\big(p_2 x_1(n+1) & -\big(p_2 x_2(n+1) & -\big(p_2 x_3(n+1) \\ +p_1 x_1(n+2)\big) & +p_1 x_2(n+2)\big) & +p_1 x_3(n+2)\big) \end{pmatrix}. \tag{2.2.13}$$

Using the properties of determinants, it follows from (2.2.13) that

$$W(n+1) = \det \begin{pmatrix} x_1(n+1) & x_2(n+1) & x_3(n+1) \\ x_1(n+2) & x_2(n+2) & x_3(n+2) \\ -p_3(n)x_1(n) & -p_3(n)x_2(n) & -p_3(n)x_3(n) \end{pmatrix} \tag{2.2.14}$$

$$= -p_3(n) \det \begin{pmatrix} x_1(n+1) & x_2(n+1) & x_3(n+1) \\ x_1(n+2) & x_2(n+2) & x_3(n_2) \\ x_1(n) & x_2(n) & x_3(n) \end{pmatrix}$$

$$= -p_3(n)(-1)^2 \det \begin{pmatrix} x_1(n) & x_2(n) & x_3(n) \\ x_1(n+2) & x_2(n+2) & x_3(n+2) \\ x_1(n+1) & x_2(n+1) & x_3(n+1) \end{pmatrix}.$$

Thus

$$W(n+1) = (-1)^3 p_3(n) W(n). \tag{2.2.15}$$

Using formula (1.2.3), the solution of (2.2.15) is given by

$$\boxed{W(n) = \left[\prod_{i=n_0}^{n-1} (-1)^3 p_3(i)\right] W(n_0) = (-1)^{3(n-n_0)} \prod_{i=n_0}^{n-1} p_3(i) W(n_0).}$$

□

This completes the proof of the lemma for $k = 3$. The general case is left to the reader as Exercises 2.2, Problem 6.

We now examine and treat one of the special cases that arises as we try to apply this Casoratian. For example, if (2.2.7) has constant coefficients $p_1, p_2, \ldots, p_k$, then we have

$$\boxed{W(n) = (-1)^{k(n-n_0)} p_k^{(n-n_0)} W(n_0).} \tag{2.2.16}$$

Formula (2.2.10) has the following important correspondence.

Corollary 2.14. *Suppose that $p_k(n) \neq 0$ for all $n \geq n_0$. Then the Casoratian $W(n) \neq 0$ for all $n \geq n_0$ if and only if $W(n_0) \neq 0$.*

Proof. This corollary follows immediately from formula (2.2.10) (Exercises 2.2, Problem 7). □

Let us have a close look at Corollary 2.14 and examine what it really says. The main point in the corollary is that either the Casoratian is identically zero (i.e., zero for all $n \geq n_0$, for some n_0) or never zero for any $n \geq n_0$. Thus to check whether $W(n) \neq 0$ for all $n \in \mathbb{Z}^+$, we need only to check whether $W(0) \neq 0$. Note that we can always choose the most suitable n_0 and compute $W(n_0)$ there.

Next we examine the relationship between the linear independence of solutions and their Casoratian. Basically, we will show that a set of k solutions is a *fundamental set* (i.e., linearly independent) if their Casoratian $W(n)$ is never zero.

To determine the preceding statement we contemplate k solutions $x_1(n), x_2(n), \ldots, x_k(n)$ of (2.2.7). Suppose that for some constants a_1, $a_2, \ldots, a_k$ and $n_0 \in \mathbb{Z}^+$,

$$a_1x_1(n) + a_2x_2(n) + \cdots + a_k(n)\ x_k(n) = 0, \qquad \text{for all } n \geq n_0.$$

Then we can generate the following $k-1$ equations:

$$\begin{aligned} a_1x_1(n+1) + a_2x_2(n+1) + \cdots + a_kx_k(n+1) &= 0,\\ &\vdots\\ a_1x_1(n+k-1) + a_2x_2(n+k-1) + \cdots + a_kx_k(n+k-1) &= 0. \end{aligned}$$

This assemblage may be transcribed as

$$X(n)\xi = 0, \tag{2.2.17}$$

where

$$X(n) = \begin{pmatrix} x_1(n) & x_2(n) & \ldots & x_k(n) \\ x_1(n+1) & x_2(n+1) & \ldots & x_k(n+1) \\ \vdots & \vdots & & \vdots \\ x_1(n+k-1) & x_2(n+k-1) & \ldots & x_k(n+k-1) \end{pmatrix},$$

$$\xi = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}.$$

Observe that $W(n) = \det X(n)$.

Linear algebra tells us that the vector (2.2.17) has only the trivial (or zero) solution (i.e., $a_1 = a_2 = \cdots = a_k = 0$) if and only if the matrix $X(n)$ is nonsingular (invertible) (i.e., det $X(n) = W(n) \neq 0$ for all $n \geq n_0$). This deduction leads us to the following conclusion.

Theorem 2.15. *The set of solutions* $x_1(n), x_2(n), \ldots, x_k(n)$ *of* (2.2.7) *is a fundamental set if and only if for some* $n_0 \in Z^+$, *the Casoratian* $W(n_0) \neq 0$.

PROOF. Exercises 2.2, Problem 8. □

Example 2.16. Verify that $\{n, 2^n\}$ is a fundamental set of solutions of the equation

$$x(n+2) - \frac{3n-2}{n-1}x(n+1) + \frac{2n}{n-1}x(n) = 0.$$

Solution We leave it to the reader to verify that n and 2^n are solutions of the equation. Now, the Casoratian of the solutions n, 2^n is given by

$$W(n) = \det\begin{pmatrix} n & 2^n \\ n+1 & 2^{n+1} \end{pmatrix}.$$

Thus

$$W(0) = \det\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} = -1 \neq 0.$$

Hence by Theorem 2.15, the solutions n, 2^n are linearly independent and thus form a fundamental set.

Example 2.17. Consider the third-order difference equation

$$x(n+3) + 3x(n+2) - 4x(n+1) - 12x(n) = 0.$$

Show that the functions 2^n, $(-2)^n$, and $(-3)^n$ form a fundamental set of solutions of the equation.

Solution

(i) Let us verify that 2^n is a legitimate solution by substituting $x(n) = 2^n$ into the equation:

$$2^{n+3} + (3)(2^{n+1}) - (4)(2^{n+1}) - (12)(2^n) = 2^n[8 + 12 - 8 - 12] = 0.$$

We leave it to the reader to verify that $(-2)^n$ and $(-3)^n$ are solutions of the equation.

(ii) To affirm the linear independence of these solutions we construct the Casoratian

$$W(n) = \det\begin{pmatrix} 2^n & (-2)^n & (-3)^n \\ 2^{n+1} & (-2)^{n+1} & (-3)^{n+1} \\ 2^{n+2} & (-2)^{n+2} & (-3)^{n+2} \end{pmatrix}.$$

Thus

$$W(0) = \det \begin{pmatrix} 1 & 1 & 1 \\ 2 & -2 & 3 \\ 4 & 4 & 9 \end{pmatrix} = -20 \neq 0.$$

By Theorem 2.15, the solutions 2^n, $(-2)^n$, and 3^n are linearly independent, and thus form a fundamental set.

We are now ready to discuss the fundamental theorem of homogeneous linear difference equations.

Theorem 2.18 (The Fundamental Theorem). *If $p_k(n) \neq 0$ for all $n \geq n_0$, then* (2.2.7) *has a fundamental set of solutions for $n \geq n_0$.*

Proof. By Theorem 2.7, there are solutions $x_1(n), x_2(n), \ldots, x_k(n)$ such that $x_i(n_0+i-1) = 1$, $x_i(n_0) = x_i(n_0+1) = \cdots = x_i(n_0+i-2) = x_i(n_0+i) = \cdots = x_i(n_0+k-1) = 0, 1 \leq i \leq k$. Hence $x_1(n_0) = 1, x_2(n_0+1) = 1, x_3(n_0+2) = 1, \ldots, x_k(n_0+k-1) = 1$. It follows that $W(n_0) = \det I = 1$. This implies by Theorem 2.15 that the set $\{x_1(n), x_2(n), \ldots, x_k(n)\}$ is a fundamental set of solutions of (2.2.7). □

We remark that there are infinitely many fundamental sets of solutions of (2.2.7). The next result presents a method of generating fundamental sets starting from a known set.

Lemma 2.19. *Let $x_1(n)$ and $x_2(n)$ be two solutions of* (2.2.7)*. Then the following statements hold:*

(i) *$x(n) = x_1(n) + x_2(n)$ is a solution of* (2.2.7)*.*

(ii) *$\tilde{x}(n) = a x_1(n)$ is a solution of* (2.2.7) *for any constant a.*

Proof. (Exercises 2.2, Problem 9.) □

From the preceding lemma we conclude the following principle.

Superposition Principle. If $x_1(n), x_2(n), \ldots, x_r(n)$ are solutions of (2.2.7), then

$$x(n) = a_1 x_1(n) + a_2 x_2(n) + \cdots + a_r x_r(n)$$

is also a solution of (2.2.7) (Exercises 2.2, Problem 12).

Now let $\{x_1(n), x_2(n), \ldots, x_k(n)\}$ be a fundamental set of solutions of (2.2.7) and let $x(n)$ be any given solution of (2.2.7). Then there are constants $a_1, a_2, \ldots, a_k$ such that $x(n) = \sum_{i=1}^{k} a_i x_i(n)$. To show this we use the notation (2.2.17) to write $X(n)\xi = \hat{x}(n)$, where

$$\hat{x}(n) = \begin{pmatrix} x(n) \\ x(n+1) \\ \vdots \\ x(n+k-1) \end{pmatrix}.$$

Since $X(n)$ is invertible (Why?), it follows that

$$\xi = X^{-1}(n)\hat{x}(n),$$

and, for $n = n_0$,

$$\xi = X^{-1}(n_0)\hat{x}(n_0).$$

The above discussion leads us to define the general solution of (2.2.7).

Definition 2.20. Let $\{x_1(n), x_2(n), \ldots, x_k(n)\}$ be a fundamental set of solutions of (2.2.7). Then the *general solution* of (2.2.7) is given by $x(n) = \sum_{i=1}^{k} a_i x_i(n)$, for arbitrary constants a_i.

It is worth noting that any solution of (2.2.7) may be obtained from the general solution by a suitable choice of the constants a_i.

The preceding results may be restated using the elegant language of linear algebra as follows: Let S be the set of all solutions of (2.2.7) with the operations $+$, $\cdot$ defined as follows:

(i) $(x + y)(n) = x(n) + y(n), \qquad \text{for } x, y \in S, \quad n \in Z^+,$

(ii) $(ax)(n) = ax(n), \qquad \text{for } x \in S, a \text{ constant}.$

Equipped with linear algebra we now summarize the results of this section in a compact form.

Theorem 2.21. *The space $(S, +, \cdot)$ is a linear (vector) space of dimension k.*

PROOF. Use Lemma 2.19. To construct a basis of S we can use the fundamental set in Theorem 2.18 (Exercises 2.2, Problem 11). □

Exercises 2.2

1. Find the Casoratian of the following functions and determine whether they are linearly dependent or independent:

   (a) $5^n, 3 \cdot 5^{n+2}, e^n$.

   (b) $5^n, n\,5^n, n^2\,5^n$.

   (c) $(-2)^n, 2^n, 3$.

   (d) $0, 3^n, 7^n$.

2. Find the Casoratian $W(n)$ of the solutions of the difference equations:

   (a) $x(n+3) - 10x(n+2) + 31x(n+1) - 30x(n) = 0$, if $W(0) = 6$.

   (b) $x(n+3) - 3x(n+2) + 4x(n+1) - 12x(n) = 0$, if $W(0) = 26$.

3. For the following difference equations and their accompanied solutions:

   (i) determine whether these solutions are linearly independent, and

(ii) find, if possible, using only the given solutions, the general solution:
(a) $x(n+3) - 3x(n+2) + 3x(n+1) - x(n) = 0; \quad 1, n, n^2$,
(b) $x(n+2) + x(n) = 0; \quad \cos\left(\frac{n\pi}{2}\right), \sin\left(\frac{n\pi}{2}\right)$,
(c) $x(n+3)+x(n+2)-8x(n+1)-12x(n) = 0; \quad 3^n, (-2)^n, (-2)^{n+3}$,
(d) $x(n+4) - 16x(n) = 0; \quad 2^n, n2^n, n^2 2^n$.

4. Verify formula (2.2.10) for the general case.

5. Show that the Casoratian $W(n)$ in formula (2.2.9) may be given by the formula

$$W(n) = \det \begin{pmatrix} x_1(n) & x_2(n) & \dots & x_k(n) \\ \Delta x_1(n) & \Delta x_2(n) & \dots & \Delta x_k(n) \\ \vdots & \vdots & & \vdots \\ \Delta^{k-1} x_1(n) & \Delta^{k-1} x_2(n) & \dots & \Delta^{k-1} x_k(n) \end{pmatrix}.$$

6. Verify formula (2.2.16).

7. Prove Corollary 2.14.

8. Prove Theorem 2.15.

9. Prove Lemma 2.19.

10. Prove the superposition principle: If $x_1(n), x_2(n), \dots, x_r$ are solutions of (2.2.7), then any linear combination of them is also a solution of (2.2.7).

11. Prove Theorem 2.21.

12. Suppose that for some integer $m \geq n_0, p_k(m) = 0$ in (2.2.1).
   (a) What is the value of the Casoratian for $n \geq m$?
   (b) Does Corollary 2.14 still hold? (Why?)

*13. Show that the equation $\Delta^2 y(n) = p(n)y(n+1)$ has a fundamental set of solutions whose Casoratian $W(n) = -1$.

14. Contemplate the second-order difference equation $u(n+2)+p_1(n)u(n+1) + p_2(n)u(n) = 0$. If $u_1(n)$ and $u_2(n)$ are solutions of the equation and $W(n)$ is their Casoratian, prove that

$$\boxed{u_2(n) = u_1(n) \left[\sum_{r=0}^{n-1} W(r)/u_1(r)u_1(r+1) \right].} \tag{2.2.18}$$

15. Contemplate the second-order difference equation $u(n+2) - \frac{(n+3)}{(n+2)} u(n+1) + \frac{2}{(n+2)} u(n) = 0$.

(a) Verify that $u_1(n) = \frac{2^n}{n!}$ is a solution of the equation.

(b) Use formula (2.2.18) to find another solution $u_2(n)$ of the equation.

16. Show that $u(n) = (n+1)$ is a solution of the equation $u(n+2) - u(n+1) - 1/(n+1)u(n) = 0$ and then find a second solution of the equation by using the method of Exercises 2.2, Problem 15.

2.3 Linear Homogeneous Equations with Constant Coefficients

Consider the kth-order difference equation

$$x(n+k) + p_1 x(n+k-1) + p_2 x(n+k-2) + \cdots + p_k x(n) = 0, \quad (2.3.1)$$

where the p_i's are constants and $p_k \neq 0$. Our objective now is to find a fundamental set of solutions and, consequently, the general solution of (2.3.1). The procedure is rather simple. We suppose that solutions of (2.3.1) are in the form λ^n, where λ is a complex number. Substituting this value into (2.3.1), we obtain

$$\lambda^k + p_1 \lambda^{k-1} + \cdots + p_k = 0. \quad (2.3.2)$$

This is called the *characteristic equation* of (2.3.1), and its roots λ are called the *characteristic roots*. Notice that since $p_k \neq 0$, none of the characteristic roots is equal to zero. (Why?) (Exercises 2.3, Problem 19.)

We have two situations to contemplate:

Case (a). Suppose that the characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct. We are now going to show that the set $\{\lambda_1^n, \lambda_2^n, \ldots, \lambda_k^n\}$ is a fundamental set of solutions. To prove this, by virtue of Theorem 2.15 it suffices to show that $W(0) \neq 0$, where $W(n)$ is the Casoratian of the solutions. That is,

$$W(0) = \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_k^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \ldots & \lambda_k^{k-1} \end{pmatrix}. \quad (2.3.3)$$

This determinant is called the *Vandermonde determinant.*

It may be shown by mathematical induction that

$$\boxed{W(0) = \prod_{1 \leq i < j \leq k} (\lambda_j - \lambda_i).} \quad (2.3.4)$$

The reader will prove this conclusion in Exercises 2.3, Problem 20.

Since all the λ_i's are distinct, it follows from (2.3.4) that $W(0) \neq 0$. This fact proves that $\{\lambda_1^n, \lambda_2^n, \ldots, \lambda_k^n\}$ is a fundamental set of solutions of (2.3.1). Consequently, the general solution of (2.3.1) is

$$x(n) = \sum_{i=1}^{k} a_i \lambda_i^n, \quad a_i \text{ a complex number.} \tag{2.3.5}$$

Case (b). Suppose that the distinct characteristic roots are $\lambda_1, \lambda_2, \ldots, \lambda_r$ with multiplicities $m_1, m_2, \ldots, m_r$ with $\sum_{i=1}^{r} m_i = k$, respectively. In this case, (2.3.1) may be written as

$$(E - \lambda_1)^{m_1}(E - \lambda_2)^{m_2} \cdots (E - \lambda_r)^{m_r} x(n) = 0. \tag{2.3.6}$$

A vital observation here is that if $\psi_1(n), \psi_2(n), \ldots, \psi_{m_i}(n)$ are solutions of

$$(E - \lambda_i)^{m_i} x(n) = 0, \tag{2.3.7}$$

then they are also solutions of (2.3.6). For if $\Psi_s(n)$ is a solution of (2.3.7), then $(E - \lambda_i)^{m_i}\Psi_s(n) = 0$. Now

$$\begin{aligned}
&(E - \lambda_1)^{m_1} \cdots (E - \lambda_i)^{m_i} \cdots (E - \lambda_r)^{m_r} \Psi_s(n) \\
&= (E - \lambda_1)^{m_1} \cdots (E - \lambda_{i-1})^{m_{i-1}} (E - \lambda_{i+1})^{m_{i+1}} \cdots \\
&(E - \lambda_r)^{m_r} (E - \lambda_i)^{m_i} \Psi_s(n) = 0.
\end{aligned}$$

Suppose we are able to find a fundamental set of solutions for each (2.3.7), $1 \leq i \leq r$. It is not unreasonable to expect, then, that the union of these r fundamental sets would be a fundamental set of solutions of (2.3.6). In the following lemma we will show that this is indeed the case.

Lemma 2.22. *The set* $G_i = \left\{\lambda_i^n, \binom{n}{1}\lambda_i^{n-1}, \binom{n}{2}\lambda_2^{n-2}, \ldots, \binom{n}{m_i-1}\lambda_i^{n-m_i+1}\right\}$ *is a fundamental set of solutions of* (2.3.7) *where* $\binom{n}{1} = n, \binom{n}{2} = \frac{n(n-1)}{2!}, \ldots, \binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!}$.

Proof. To show that G_i is a fundamental set of solutions of (2.3.7), it suffices, by virtue of Corollary 2.14, to show that $W(0) \neq 0$. But

$$W(0) = \begin{vmatrix}
1 & 0 & \ldots & 0 \\
\lambda_i & 1 & \ldots & 0 \\
\lambda_i^2 & 2\lambda_i & \ldots & 0 \\
\vdots & \vdots & & \vdots \\
\lambda_i^{m_i-1} & \dfrac{(m_i-1)}{1!}\lambda_i^{m_i-2} & \ldots & \dfrac{1}{2!3!\cdots(m_i-2)!}
\end{vmatrix}.$$

Hence

$$W(0) = \frac{1}{(2!3!\ldots(m_i-2)!} \neq 0.$$

It remains to show that $\binom{n}{r}\lambda_i^{n-r}$ is a solution of (2.3.7). From equation (2.1.9) it follows that

$$\begin{aligned}(E-\lambda_i)^{m_i}\binom{n}{r}\lambda_i^{n-r} &= \lambda_i^{n-r}(\lambda_i E-\lambda_i)^{m_i}\binom{n}{r}\\ &= \lambda_i^{n+m_i-r}(E-I)^{m_i}\binom{n}{r}\\ &= \lambda_i^{n+m_i-r}\Delta^{m_i}\binom{n}{r}\\ &= 0 \quad \text{using (2.1.6).}\end{aligned}$$ □

Now we are finally able to find a fundamental set of solutions.

Theorem 2.23. *The set $G = \bigcup_{i=1}^{r} G_i$ is a fundamental set of solutions of* (2.3.6)*.*

PROOF. By Lemma 2.22, the functions in G are solutions of (2.3.6). Now

$$W(0) = \det\begin{pmatrix} 1 & 0 & \dots & 1 & 0 & \dots\\ \lambda_1 & 1 & \dots & \lambda_r & 1 & \dots\\ \lambda_1^2 & 2\lambda_1 & \dots & \lambda_r^2 & 2\lambda_r & \dots\\ \vdots & \vdots & & \vdots & \vdots & \\ \lambda_1^{k-1} & (k-1)\lambda_1^{k-2} & \dots & \lambda_r^{k-1} & (k-1)\lambda_r^{k-2} & \dots \end{pmatrix}. \tag{2.3.8}$$

This determinant is called the generalized Vandermonde determinant. (See Appendix B.) It may be shown [76] that

$$\boxed{W(0) = \prod_{1\le i<j\le k} (\lambda_j-\lambda_i)^{m_j m_i}.} \tag{2.3.9}$$

As $\lambda_i \neq \lambda_j$, $W(0) \neq 0$. Hence by Corollary 2.14 the Casoratian $W(n) \neq 0$ for all $n \geq 0$. Thus by Theorem 2.15, G is a fundamental set of solutions. □

Corollary 2.24. *The general solution of* (2.3.6) *is given by*

$$x(n) = \sum_{i=1}^{r} \lambda_i^n \left(a_{i0} + a_{i1}n + a_{i2}n^2 + \cdots + a_{i,m_i-1}n^{m_i-1}\right). \tag{2.3.10}$$

PROOF. Use Lemma 2.22 and Theorem 2.23. □

Example 2.25. Solve the equation

$$x(n+3) - 7x(n+2) + 16x(n+1) - 12x(n) = 0,$$
$$x(0) = 0, \quad x(1) = 1, \quad x(2) = 1.$$

Solution The characteristic equation is

$$r^3 - 7r^2 + 16r - 12 = 0.$$

Thus, the characteristic roots are $\lambda_1 = 2 = \lambda_2, \lambda_3 = 3$.

The characteristic roots give us the general solution

$$x(n) = a_0 2^n + a_1 n 2^n + b_1 3^n.$$

To find the constants a_0, a_1, and b_1, we use the initial data

$$\begin{aligned} x(0) &= a_0 + b_1 = 0, \\ x(1) &= 2a_0 + 2a_1 + 3b_1 = 1, \\ x(2) &= 4a_0 + 8a_1 + 9b_1 = 1. \end{aligned}$$

Finally, after solving the above system of equations, we obtain

$$a_0 = 3, \quad a_1 = 2, \quad b_1 = -3.$$

Hence the solution of the equation is given by $x(n) = 3(2^n) + 2n(2^n) - 3^{n+1}$.

Example 2.26. Complex Characteristic Roots

Suppose that the equation $x(n+2) + p_1 x(n+1) + p_2 x(n) = 0$ has the complex roots $\lambda_1 = \alpha + i\beta, \lambda_2 = \alpha - i\beta$. Its general solution would then be

$$x(n) = c_1(\alpha + i\beta)^n + c_2(\alpha - i\beta)^n.$$

Recall that the point (α, β) in the complex plane corresponds to the complex number $\alpha + i\beta$. In polar coordinates,

$$\alpha = r\cos\theta, \qquad \beta = r\sin\theta, \qquad r = \sqrt{\alpha^2 + \beta^2}, \qquad \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right).$$

Hence,[5]

$$\begin{aligned} x(n) &= c_1(r\cos\theta + ir\sin\theta)^n + c_2(r\cos\theta - ir\sin\theta)^n \\ &= r^n[(c_1 + c_2)\cos(n\theta) + i(c_1 - c_2)\sin(n\theta)] \\ &= r^n[a_1\cos(n\theta) + a_2\sin(n\theta)], \end{aligned} \tag{2.3.11}$$

where $a_1 = c_1 + c_2$ and $a_2 = i(c_1 - c_2)$.

Let

$$\cos\omega = \frac{a_1}{\sqrt{a_1^2 + a_2^2}}, \quad \sin\omega = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}, \quad \omega = \tan^{-1}\left(\frac{a_2}{a_1}\right).$$

[5]We used De Moivre's Theorem: $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$.

Then (2.3.11) becomes

$$\begin{aligned} x(n) &= r^n\sqrt{a_1^2+a_2^2}[\cos\omega\cos(n\theta)+\sin\omega\sin(n\theta)] \\ &= r^n\sqrt{a_1^2+a_2^2}\cos(n\theta-\omega), \\ x(n) &= Ar^n\cos(n\theta-\omega). \end{aligned} \tag{2.3.12}$$

Example 2.27. The Fibonacci Sequence (The Rabbit Problem)

This problem first appeared in 1202, in *Liber abaci*, a book about the abacus, written by the famous Italian mathematician Leonardo di Pisa, better known as Fibonacci. The problem may be stated as follows: How many pairs of rabbits will there be after one year if starting with one pair of mature rabbits, if each pair of rabbits gives birth to a new pair each month starting when it reaches its maturity age of two months? (See Figure 2.1.)

Table 2.2 shows the number of pairs of rabbits at the end of each month. The first pair has offspring at the end of the first month, and thus we have two pairs. At the end of the second month only the first pair has offspring, and thus we have three pairs. At the end of the third month, the first and second pairs will have offspring, and hence we have five pairs. Continuing this procedure, we arrive at Table 2.2. If $F(n)$ is the number of pairs of rabbits at the end of n months, then the recurrence relation that represents this model is given by the second-order linear difference equation

$$F(n+2) = F(n+1) + F(n), \qquad F(0) = 1, \qquad F(1) = 2, \qquad 0 \le n \le 10.$$

This example is a special case of the Fibonacci sequence, given by

$$F(n+2) = F(n+1) + F(n), \qquad F(0) = 0, \qquad F(1) = 1, \qquad n \ge 0. \tag{2.3.13}$$

The first 14 terms are given by 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, and 377, as already noted in the rabbit problem.

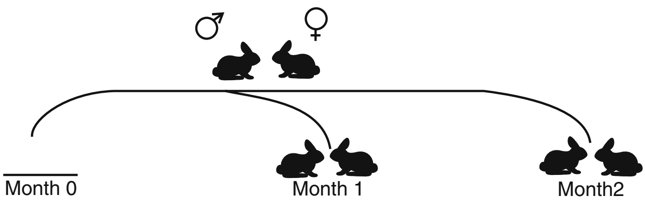


FIGURE 2.1.

TABLE 2.2. Rabbits' population size.

Month	0	1	2	3	4	5	6	7	8	9	10	11	12
Pairs	1	2	3	5	8	13	21	34	55	89	144	233	377

The characteristic equation of (2.3.13) is

$$\lambda^2 - \lambda - 1 = 0.$$

Hence the characteristic roots are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. The general solution of (2.3.13) is

$$F(n) = a_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + a_2 \left(\frac{1-\sqrt{5}}{2}\right)^n, \qquad n \geq 1. \tag{2.3.14}$$

Using the initial values $F(1) = 1$ and $F(2) = 1$, one obtains

$$a_1 = \frac{1}{\sqrt{5}}, \qquad a_2 = -\frac{1}{\sqrt{5}}.$$

Consequently,

$$F(n) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right] = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n). \tag{2.3.15}$$

It is interesting to note that $\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = \alpha \approx 1.618$ (Exercises 2.3, Problem 15). This number is called the *golden mean*, which supposedly represents the ratio of the sides of a rectangle that is most pleasing to the eye. This Fibonacci sequence is very interesting to mathematicians; in fact, an entire publication, *The Fibonacci Quarterly*, dwells on the intricacies of this fascinating sequence.

Exercises 2.3

1. Find homogeneous difference equations whose solutions are:
   (a) $2^{n-1} - 5^{n+1}$.
   (b) $3\cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right)$.
   (c) $(n+2)5^n \sin\left(\frac{n\pi}{4}\right)$.
   (d) $(c_1 + c_2 n + c_3 n^2)7^n$.
   (e) $1 + 3n - 5n^2 + 6n^3$.
2. Find a second-order linear homogeneous difference equation that generates the sequence 1, 2, 5, 12, 29, ...; then write the solution of the obtained equation.

In each of Problems 3 through 8, write the general solution of the difference equation.

3. $x(n+2) - 16x(n) = 0$.
4. $x(n+2) + 16x(n) = 0$.
5. $(E-3)^2(E^2+4)x(n) = 0$.

6. $\Delta^3 x(n) = 0$.

7. $(E^2 + 2)^2 x(n) = 0$.

8. $x(n+2) - 6x(n+1) + 14x(n) = 0$.

9. Consider Example 2.26. Verify that $x_1(n) = r^n \cos n\theta$ and $x_2(n) = r^n \sin n\theta$ are two linearly independent solutions of the given equation.

10. Consider the integral defined by

$$I_k(\varphi) = \int_0^\pi \frac{\cos(k\theta) - \cos(k\varphi)}{\cos\theta - \cos\varphi} d\theta, \quad k = 0, 1, 2, \ldots, \quad \varphi \in \mathbb{R}.$$

   (a) Show that $I_k(\varphi)$ satisfies the difference equation

$$I_{n+2}(\varphi) - 2\cos\varphi I_{n+1}(\varphi) + I_n(\varphi) = 0, \qquad I_0(\varphi) = 0, \qquad I_1(\varphi) = \pi.$$

   (b) Solve the difference equation in part (a) to find $I_n(\varphi)$.

11. The Chebyshev polynomials of the first and second kinds are defined, respectively, as follows:

$$T_n(x) = \cos(n \cos^{-1}(x)), \qquad U_n(x) = \frac{1}{\sqrt{1-x^2}} \sin[(n+1)\cos^{-1}(x)],$$

for $|x| < 1$.

   (a) Show that $T_n(x)$ obeys the difference equation

$$T_{n+2}(x) - 2xT_{n+1}(x) + T_n(x) = 0, \qquad T_0(x) = 1, \qquad T_1(x) = x.$$

   (b) Solve the difference equation in part (a) to find $T_n(x)$.

   (c) Show that $U_n(x)$ satisfies the difference equation

$$U_{n+2}(x) - 2xU_{n+1}(x) + U_n(x) = 0, \qquad U_0(x) = 1, \qquad U_1(x) = 2x.$$

   (d) Write down the first three terms of $T_n(x)$ and $U_n(x)$.

   (e) Show that $T_n(\cos\theta) = \cos n\theta$ and that

$$U_n(\cos\theta) = (\sin[(n+1)\theta])/\sin\theta.$$

12. Show that the general solution of

$$x(n+2) - 2sx(n+1) + x(n) = 0, \qquad |s| < 1,$$

is given by

$$x(n) = c_1 T_n(s) + c_2 U_n(s).$$

13. Show that the general solution of $x(n+2) + p_1 x(n+1) + p_2 x(n) = 0$, $p_2 > 0$, $p_1^2 < 4p_2$, is given by $x(n) = r^n [c_1 T_n(s) + c_2 U_{n-1}(s)]$, where $r = \sqrt{p_2}$ and $s = P_1/(2\sqrt{p_2})$.

14. The Lucas numbers L_n are defined by the difference equation

$$L_{n+2} = L_{n+1} + L_n, \quad n \geq 0, \quad L_0 = 2, \quad L_1 = 1.$$

Solve the difference equation to find L_n.

15. Show that $\lim_{n\to\infty}(F(n+1))/F(n) = \alpha$, where $\alpha = (1+\sqrt{5})/2$.

16. Prove that consecutive Fibonacci numbers $F(n)$ and $F(n+1)$ are relatively prime.

17. (a) Prove that $F(n)$ is the nearest integer to $1/\sqrt{5}((1+\sqrt{5})/2)^n$.

    (b) Find $F(17)$, $F(18)$, and $F(19)$, applying part (a).

*18. Define $x = a \bmod p$ if $x = mp + a$. Let p be a prime number with $p > 5$.

    (a) Show that $F(p) = 5^{(p-1)/2} \bmod p$.

    (b) Show that $F(p) = \pm 1 \bmod p$.

19. Show that if $p_k \neq 0$ in (2.3.1), then none of its characteristic roots is equal to zero.

20. Show that the Vandermonde determinant (2.3.3) is equal to

$$\prod_{1\leq i<j\leq k} (\lambda_j - \lambda_i).$$

21. Find the value of the $n \times n$ tridiagonal determinant

$$D(n) = \begin{vmatrix} b & a & 0 & \dots & 0 & 0 \\ a & b & a & \dots & 0 & 0 \\ 0 & a & b & \dots & 0 & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & b & a \\ 0 & 0 & 0 & \dots & a & b \end{vmatrix}.$$

22. Find the value of the $n \times n$ tridiagonal determinant

$$D(n) = \begin{vmatrix} a & b & 0 & \dots & 0 & 0 \\ c & a & b & \dots & 0 & 0 \\ 0 & c & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & b \\ 0 & 0 & 0 & \dots & c & a \end{vmatrix}.$$

2.4 Linear Nonhomogeneous Equations: Method of Undetermined Coefficients

In the last two sections we developed the theory of linear homogeneous difference equations. Moreover, in the case of equations with constant coefficients we have shown how to construct their solutions. In this section we focus our attention on solving the kth-order linear nonhomogeneous equation

$$y(n+k) + p_1(n)y(n+k-1) + \cdots + p_k(n)y(n) = g(n), \tag{2.4.1}$$

where $p_k(n) \neq 0$ for all $n \geq n_0$. The sequence $g(n)$ is called the *forcing term*, the *external force*, the *control*, or the *input* of the system. As we will discuss later in Chapter 6, equation (2.4.1) represents a physical system in which $g(n)$ is the input and $y(n)$ is the output (Figure 2.2). Thus solving (2.4.1) amounts to determining the output $y(n)$ given the input $g(n)$. We may look at $g(n)$ as a control term that the designing engineer uses to force the system to behave in a specified way.

Before proceeding to present general results concerning (2.4.1) we would like to raise the following question: Do solutions of (2.4.1) form a vector space? In other words, is the sum of two solutions of (2.4.1) a solution of (2.4.1)? And is a multiple of a solution of (2.4.1) a solution of (2.4.1)? Let us answer these questions through the following example.

Example 2.28. Contemplate the equation

$$y(n+2) - y(n+1) - 6y(n) = 5(3^n).$$

(a) Show that $y_1(n) = n(3^{n-1})$ and $y_2(n) = (1+n)3^{n-1}$ are solutions of the equation.

(b) Show that $y(n) = y_2(n) - y_1(n)$ is not a solution of the equation.

(c) Show that $\varphi(n) = cn(3^{n-1})$ is not a solution of the equation, where c is a constant.

Solution

(a) The verification that y_1 and y_2 are solutions is left to the reader.

(b) $y(n) = y_2(n) - y_1(n) = 3^{n-1}$. Substituting this into the equation yields

$$3^{n+1} - 3^n - 63^{n-1} = 3^n[3 - 1 - 2] = 0 \neq 5(3^n).$$

(c) By substituting for $\varphi(n)$ into the equation we see easily that $\varphi(n)$ is not a solution.

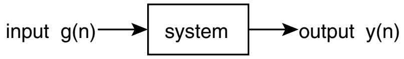


FIGURE 2.2. Input–output system.

Conclusion

(i) From the above example we conclude that in contrast to the situation for homogeneous equations, solutions of the nonhomogeneous equation (2.4.1) do not form a vector space. In particular, neither the sum (difference) of two solutions nor a multiple of a solution is a solution.

(ii) From part (b) in Example 2.28 we found that the difference of the solutions $y_2(n)$ and $y_1(n)$ of the nonhomogeneous equation is actually a solution of the associated homogeneous equation. This is indeed true for the general nth-order equation, as demonstrated by the following result.

Theorem 2.29. *If $y_1(n)$ and $y_2(n)$ are solutions of* (2.4.1)*, then $x(n) = y_1(n) - y_2(n)$ is a solution of the corresponding homogeneous equation*

$$x(n+k) + p_1(n)x(n+k-1) + \cdots + p_k(n)x(n) = 0. \tag{2.4.2}$$

PROOF. The reader will undertake the justification of this theorem in Exercises 2.4, Problem 12. □

It is customary to refer to the general solution of the homogeneous equation (2.4.2) as the *complementary solution* of the nonhomogeneous equation (2.4.1), and it will be denoted by $y_c(n)$. A solution of the nonhomogeneous equation (2.4.1) is called a *particular solution* and will be denoted by $y_p(n)$. The next result gives us an algorithm to generate all solutions of the nonhomogeneous equation (2.4.1).

Theorem 2.30. *Any solution $y(n)$ of* (2.4.1) *may be written as*

$$\boxed{y(n) = y_p(n) + \sum_{i=1}^{k} a_i x_i(n),}$$

where $\{x_1(n), x_2(n), \ldots, x_k(n)\}$ is a fundamental set of solutions of the homogeneous equation (2.4.2).

PROOF. Observe that according to Theorem 2.29, $y(n) - y_p(n)$ is a solution of the homogeneous equation (2.4.2). Thus $y(n) - y_p(n) = \sum_{i=1}^{k} a_i x_i(n)$, for some constants a_i.

The preceding theorem leads to the definition of the *general solution* of the nonhomogeneous equation (2.4.1) as

$$y(n) = y_c(n) + y_p(n). \tag{2.4.3}$$

□

We now turn our attention to finding a particular solution y_p of nonhomogeneous equations with constant coefficients such as

$$y(n+k) + p_1 y(n+k-1) + \cdots + p_k y(n) = g(n). \tag{2.4.4}$$

Because of its simplicity, we use the method of *undetermined coefficients* to compute y_p.

Basically, the method consists in making an intelligent guess as to the form of the particular solution and then substituting this function into the difference equation. For a completely arbitrary nonhomogeneous term $g(n)$, this method is not effective. However, definite rules can be established for the determination of a particular solution by this method if $g(n)$ is a linear combination of terms, each having one of the forms

$$a^n, \quad \sin(bn), \quad \cos(bn), \quad \text{or } n^k, \tag{2.4.5}$$

or products of these forms such as

$$a^n \sin(bn), \qquad a^n n^k, \qquad a^n n^k \cos(bn), \ldots . \tag{2.4.6}$$

Definition 2.31. A polynomial operator $N(E)$, where E is the shift operator, is said to be an *annihilator* of $g(n)$ if

$$N(E)g(n) = 0. \tag{2.4.7}$$

In other words, $N(E)$ is an annihilator of $g(n)$ if $g(n)$ is a solution of (2.4.7). For example, an annihilator of $g(n) = 3^n$ is $N(E) = E - 3$, since $(E-3)y(n) = 0$ has a solution $y(n) = 3^n$. An annihilator of $g(n) = \cos \frac{n\pi}{2}$ is $N(E) = E^2 + 1$, since $(E^2+1)y(n) = 0$ has a solution $y(n) = \cos \frac{n\pi}{2}$. Let us now rewrite (2.4.4) using the shift operator E as

$$p(E)y(n) = g(n), \tag{2.4.8}$$

where $p(E) = E^k + p_1 E^{k-1} + p_2 E^{k-2} + \cdots + p_k I$.

Assume now that $N(E)$ is an annihilator of $g(n)$ in (2.4.8). Applying $N(E)$ on both sides of (2.4.8) yields

$$N(E)p(E)y(n) = 0. \tag{2.4.9}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the characteristic roots of the homogeneous equation

$$p(E)y(n) = 0, \tag{2.4.10}$$

and let $\mu_1, \mu_2, \ldots, \mu_k$ be the characteristic roots of

$$N(E)y(n) = 0. \tag{2.4.11}$$

We must consider two separate cases.

Case 1. None of the λ_i's equals any of the μ_i's. In this case, write $y_p(n)$ as the general solution of (2.4.11) with undetermined constants. Substituting back this "guesstimated" particular solution into (2.4.4), we find the values of the constants. Table 2.3 contains several types of functions $g(n)$ and their corresponding particular solutions.

Case 2. $\lambda_i = \mu_j$ for some i, j. In this case, the set of characteristic roots of (2.4.9) is equal to the union of the sets $\{\lambda_i\}$, $\{\mu_j\}$ and, consequently,

TABLE 2.3. Particular solutions $y_p(n)$.

$g(n)$	$y_p(n)$
a^n	$c_1 a^n$
n^k	$c_0 + c_1 n + \cdots + c_k n^k$
$n^k a^n$	$c_0 a^n + c_1 n a^n + \cdots + c_k n^k a^n$
$\sin bn, \cos bn$	$c_1 \sin bn + c_2 \cos bn$
$a^n \sin bn, a^n \cos bn$	$(c_1 \sin bn + c_2 \cos bn) a^n$
$a^n n^k \sin bn, a^n n^k \cos bn$	$(c_0 + c_1 n + \cdots + c_k n^k) a^n \sin(bn)$ $+ (d_0 + d_1 n + \cdots d_k n^k) a^n \cos(bn)$

contains roots of higher multiplicity than the two individual sets of characteristic roots. To determine a particular solution $y_p(n)$, we first find the general solution of (2.4.9) and then drop all the terms that appear in $y_c(n)$. Then proceed as in Case 1 to evaluate the constants.

Example 2.32. Solve the difference equation

$$y(n+2) + y(n+1) - 12y(n) = n2^n. \qquad (2.4.12)$$

Solution The characteristic roots of the homogeneous equation are $\lambda_1 = 3$ and $\lambda_2 = -4$.

Hence,

$$y_c(n) = c_1 3^n + c_2(-4)^n.$$

Since the annihilator of $g(n) = n2^n$ is given by $N(E) = (E-2)^2$ (Why?), we know that $\mu_1 = \mu_2 = 2$. This equation falls in the realm of Case 1, since $\lambda_i \neq \mu_j$, for any i, j. So we let

$$y_p(n) = a_1 2^n + a_2 n 2^n.$$

Substituting this relation into equation (2.4.12) gives

$$a_1 2^{n+2} + a_2(n+2)2^{n+2} + a_1 2^{n+1} + a_2(n+1)2^{n+1} - 12a_1 2^n - 12a_2 n 2^n = n2^n,$$
$$(10a_2 - 6a_1)2^n - 6a_2 n 2^n = n2^n.$$

Hence

$$10a_2 - 6a_1 = 0 \quad \text{and} \quad -6a_2 = 1,$$

or

$$a_1 = \frac{-5}{18}, \qquad a_2 = \frac{-1}{6}.$$

The particular solution is

$$y_p(n) = \frac{-5}{18} 2^n - \frac{1}{6} n 2^n,$$

and the general solution is

$$y(n) = c_1 3^n + c_2(-4)^n - \frac{5}{18} 2^n - \frac{1}{6} n 2^n.$$

Example 2.33. Solve the difference equation

$$(E-3)(E+2)y(n) = 5(3^n). \tag{2.4.13}$$

Solution The annihilator of $5(3^n)$ is $N(E) = (E-3)$. Hence, $\mu_1 = 3$. The characteristic roots of the homogeneous equation are $\lambda_1 = 3$ and $\lambda_2 = -2$. Since $\lambda_1 = \mu_1$, we apply the procedure for Case 2.

Thus,

$$(E-3)^2(E+2)y(n) = 0. \tag{2.4.14}$$

Now $y_c(n) = c_1 3^n + c_2(-2)^n$.

We now know that the general solution of (2.4.14) is given by

$$\tilde{y}(n) = (a_1 + a_2 n)3^n + a_3(-2)^n.$$

Omitting from $\tilde{y}(n)$ the terms 3^n and $(-2)^n$ that appeared in $y_c(n)$, we set $y_p(n) = a_2 n 3^n$. Substituting this $y_p(n)$ into (2.4.13) gives

$$a_2(n+2)3^{n+2} - a_2(n+1)3^{n+1} + 6a_2 n 3^n = 5.3^n,$$

or

$$a_2 = \frac{1}{3}.$$

Thus $y_p(n) = n3^{n-1}$, and the general solution of (2.4.13) is

$$y(n) = c_1 3^n + c_2(-2)^n + n3^{n-1}.$$

Example 2.34. Solve the difference equation

$$y(n+2) + 4y(n) = 8(2^n)\cos\left(\frac{n\pi}{2}\right). \tag{2.4.15}$$

Solution The characteristic equation of the homogeneous equation is

$$\lambda^2 + 4 = 0.$$

The characteristic roots are

$$\lambda_1 = 2i, \qquad \lambda_2 = -2i.$$

Thus $r = 2, \theta = \pi/2$, and

$$y_c(n) = 2^n\left(c_1\cos\left(\frac{n\pi}{2}\right) + c_2\sin\left(\frac{n\pi}{2}\right)\right).$$

Notice that $g(n) = 2^n\cos\left(\frac{n\pi}{2}\right)$ appears in $y_c(n)$. Using Table 2.3, we set

$$y_p(n) = 2^n\left(an\cos\left(\frac{n\pi}{2}\right) + bn\sin\left(\frac{n\pi}{2}\right)\right). \tag{2.4.16}$$

Substituting (2.4.16) into (2.4.15) gives

$$\begin{aligned}2^{n+2}&\left[a(n+2)\cos\left(\frac{n\pi}{2}+\pi\right) + b(n+2)\sin\left(\frac{n\pi}{2}+\pi\right)\right]\\ &+ (4)2^n\left[an\cos\left(\frac{n\pi}{2}\right) + bn\sin\left(\frac{n\pi}{2}\right)\right] = 8(2^n)\cos\left(\frac{n\pi}{2}\right).\end{aligned}$$

Replacing $\cos((n\pi)/2 + \pi)$ by $-\cos((n\pi)/2)$, and $\sin((n\pi)/2 + \pi)$ by $-\sin((n\pi)/2)$ and then comparing the coefficients of the cosine terms leads us to $a = -1$. Then by comparing the coefficients of the sine terms, we realize that $b = 0$.

By substituting these values back into (2.4.16), we know that

$$y_p(n) = -2^n n \cos\left(\frac{n\pi}{2}\right),$$

and the general solution of (2.4.15), arrived at by adding $y_c(n)$ and $y_p(n)$, is

$$y(n) = 2^n \left(c_1 \cos \frac{n\pi}{2} + c_2 \sin\left(\frac{n\pi}{2}\right) - n\cos\left(\frac{n\pi}{2}\right)\right).$$

Exercises 2.4.

For Problems 1 through 6, find a particular solution of the difference equation.

1. $y(n+2) - 5y(n+1) + 6y(n) = 1 + n.$
2. $y(n+2) + 8y(n+1) + 12y(n) = e^n.$
3. $y(n+2) - 5y(n+1) + 4y(n) = 4^n - n^2.$
4. $y(n+2) + 8y(n+1) + 7y(n) = ne^n.$
5. $y(n+2) - y(n) = n\cos\left(\frac{n\pi}{2}\right).$
6. $(E^2 + 9)^2 y(n) = \sin\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right).$

For Problems 7 through 9 find the solution of the difference equation.

7. $\Delta^2 y(n) = 16, \qquad y(0) = 2, \qquad y(1) = 3.$
8. $\Delta^2 y(n) + 7y(n) = 2\sin\left(\frac{n\pi}{4}\right), \qquad y(0) = 0, \qquad y(1) = 1.$
9. $(E-3)(E^2+1)y(n) = 3^n, \qquad y(0) = 0, \qquad y(1) = 1, \qquad y(2) = 3.$

For Problems 10 and 11 find the general solution of the difference equation.

10. $y(n+2) - y(n) = n2^n \sin\left(\frac{n\pi}{2}\right).$
11. $y(n+2) + 8y(n+1) + 7y(n) = n2^n.$
12. Prove Theorem 2.29.
13. Consider the difference equation $y(n+2)+p_1y(n+1)+p_2y(n) = g(n)$, where $p_1^2 < 4p_2$ and $0 < p_2 < 1$. Show that if y_1 and y_2 are two solutions of the equation, then $y_1(n) - y_2(n) \to 0$ as $n \to \infty$.
14. Determine the general solution of $y(n+2)+\lambda^2 y(n) = \sum_{m=1}^{N} a_m \sin(m\pi n)$, where $\lambda > 0$ and $\lambda \neq m\pi, m = 1, 2, \ldots, N$.

15. Solve the difference equation

$$y(n+2) + y(n) = \begin{cases} 1 & \text{if } 0 \leq n \leq 2, \\ -1 & \text{if } n > 2, \end{cases}$$

with $y(0) = 0$ and $y(1) = 1$.

2.4.1 The Method of Variation of Constants (Parameters)

Contemplate the second-order nonhomogeneous difference equation

$$y(n+2) + p_1(n)y(n+1) + p_2(n)y(n) = g(n) \tag{2.4.17}$$

and the corresponding homogeneous difference equation

$$y(n+2) + p_1(n)y(n+1) + p_2(n)y(n) = 0. \tag{2.4.18}$$

The method of variation of constants is commonly used to find a particular solution $y_p(n)$ of (2.4.17) when the coefficients $p_1(n)$ and $p_2(n)$ are not constants. The method assumes that a particular solution of (2.4.17) may be written in the form

$$y(n) = u_1(n)y_1(n) + u_2(n)y_2(n), \tag{2.4.19}$$

where $y_1(n)$ and $y_2(n)$ are two linearly independent solutions of the homogeneous equation (2.4.18), and $u_1(n), u_2(n)$ are sequences to be determined later.

16. (a) Show that

$$\begin{aligned} y(n+1) = {} & u_1(n)y_1(n+1) + u_2(n)y_2(n+1) \\ & + \Delta u_1(n)y_1(n+1) + \Delta u_2(n)y_2(n+1). \end{aligned} \tag{2.4.20}$$

(b) The method stipulates that

$$\Delta u_1(n)y_1(n+1) + \Delta u_2(n)y_2(n+1) = 0. \tag{2.4.21}$$

Use (2.4.20) and (2.4.21) to show that

$$\begin{aligned} y(n+2) = {} & u_1(n)y_1(n+2) + u_2(n)y_2(n+2) \\ & + \Delta u_1(n)y_1(n+2) + \Delta u_2(n)y_2(n+2). \end{aligned}$$

(c) By substituting the above expressions for $y(n)$, $y(n+1)$, and $y(n+2)$ into (2.4.17), show that

$$\Delta u_1(n)y_1(n+2) + \Delta u_2(n)y_2(n+2) = g(n). \tag{2.4.22}$$

(d) Using expressions (2.4.21) and (2.4.22), show that

$$\Delta u_1(n) = \frac{-g(n)y_2(n+1)}{W(n+1)}, \qquad u_1(n) = \sum_{r=0}^{n-1} \frac{-g(r)y_2(r+1)}{W(r+1)}, \tag{2.4.23}$$

$$\Delta u_2(n) = \frac{g(n)y_1(n+1)}{W(n+1)}, \qquad u_2(n) = \sum_{r=0}^{n-1} \frac{g(r)y_1(r+1)}{W(r+1)}, \tag{2.4.24}$$

where $W(n)$ is the Casoratian of $y_1(n)$ and $y_2(n)$.

17. Use formulas (2.4.23) and (2.4.24) to solve the equation

$$y(n+2) - 7y(n+1) + 6y(n) = n.$$

18. Use the variation of constants method to solve the initial value problem

$$y(n+2) - 5y(n+1) + 6y(n) = 2^n, \qquad y(1) = y(2) = 0.$$

19. Use Problem 16(d) to show that the unique solution of (2.4.17) with $y(0) = y(1) = 0$ is given by

$$y(n) = \sum_{r=0}^{n-1} \frac{y_1(r+1)y_2(n) - y_2(r+1)y_1(n)}{W(r+1)}.$$

20. Consider the equation

$$x(n+1) = ax(n) + f(n). \tag{2.4.25}$$

(a) Show that

$$x(n) = a^n \left[x(0) + \frac{f(0)}{a} + \frac{f(1)}{a^2} + \cdots + \frac{f(n-1)}{a^n} \right] \tag{2.4.26}$$

is a solution of (2.4.25).

(b) Show that if $|a| < 1$ and $\{f(n)\}$ is a bounded sequence, i.e., $|f(n)| \leq M$, for some $M > 0$, $n \in \mathbb{Z}^+$, then all solutions of (2.4.25) are bounded.

(c) Suppose that $a > 1$ and $\{f(n)\}$ is bounded on $\mathbb{Z}^+$. Show that if we choose

$$x(0) = -\left(\frac{f(0)}{a} + \frac{f(1)}{a^2} + \cdots + \frac{f(n)}{a^{n+1}} + \cdots \right) = -\sum_{i=0}^{\infty} \frac{f(i)}{a^{i+1}}, \tag{2.4.27}$$

then the solution $x(n)$ given by (2.4.26) is bounded on $\mathbb{Z}^+$. Give an explicit expression for $x(n)$ in this case.

(d) Under the assumptions of part (c), show that for any choice of $x(0)$, excepting that value given by (2.4.27), the solution of (2.4.25) is unbounded.

2.5 Limiting Behavior of Solutions

To simplify our exposition we restrict our discussion to the second-order difference equation

$$y(n+2) + p_1 y(n+1) + p_2 y(n) = 0. \tag{2.5.1}$$

Suppose that λ_1 and λ_2 are the characteristic roots of the equation. Then we have the following three cases:

(a) λ_1 and λ_2 are distinct real roots. Then $y_1(n) = \lambda_1^n$ and $y_2(n) = \lambda_2^n$ are two linearly independent solutions of (2.5.1). If $|\lambda_1| > |\lambda_2|$, then we call $y_1(n)$ the *dominant solution*, and λ_1 the *dominant characteristic root*. Otherwise, $y_2(n)$ is the dominant solution, and λ_2 is the dominant characteristic root. We will now show that the limiting behavior of the general solution $y(n) = a_1\lambda_1^n + a_2\lambda_2^n$ is determined by the behavior of the dominant solution. So assume, without loss of generality, that $|\lambda_1| > |\lambda_2|$. Then

$$y(n) = \lambda_1^n \left[a_1 + a_2 \left(\frac{\lambda_2}{\lambda_1} \right)^n \right].$$

Since

$$\left| \frac{\lambda_2}{\lambda_1} \right| < 1,$$

it follows that

$$\left(\frac{\lambda_2}{\lambda_1} \right)^n \to 0 \quad \text{as } n \to \infty.$$

Consequently, $\lim_{n\to\infty} y(n) = \lim_{n\to\infty} a_1\lambda_1^n$. There are six different situations that may arise here depending on the value of λ_1 (see Figure 2.3).

1. $\lambda_1 > 1$: The sequence $\{a_1\lambda_1^n\}$ diverges to ∞ (unstable system).
2. $\lambda_1 = 1$: The sequence $\{a_1\lambda_1^n\}$ is a constant sequence.
3. $0 < \lambda_1 < 1$: The sequence $\{a_1\lambda_1^n\}$ is monotonically decreasing to zero (stable system).
4. $-1 < \lambda_1 < 0$: The sequence $\{a_1\lambda_1^n\}$ is oscillating around zero (i.e., alternating in sign) and converging to zero (stable system).
5. $\lambda_1 = -1$: The sequence $\{a_1\lambda_1^n\}$ is oscillating between two values a_1 and $-a_1$.
6. $\lambda_1 < -1$: The sequence $\{a_1\lambda_1^n\}$ is oscillating but increasing in magnitude (unstable system).

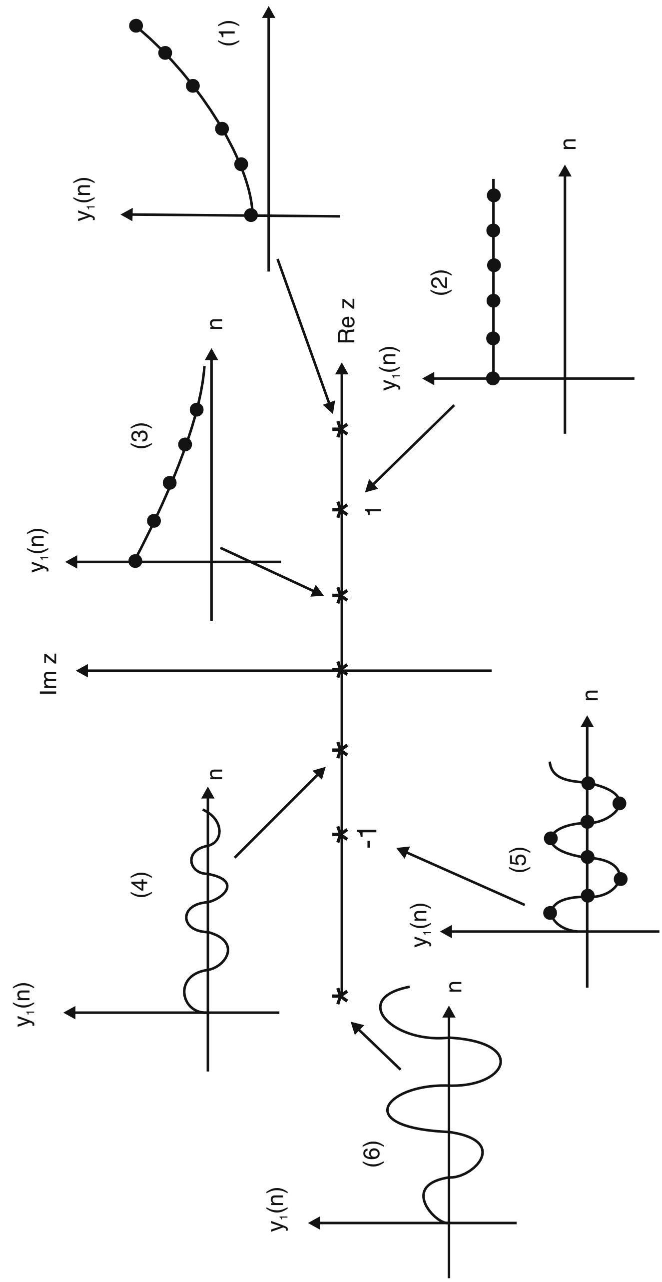


FIGURE 2.3. $(n, y(n))$ diagrams for real roots.

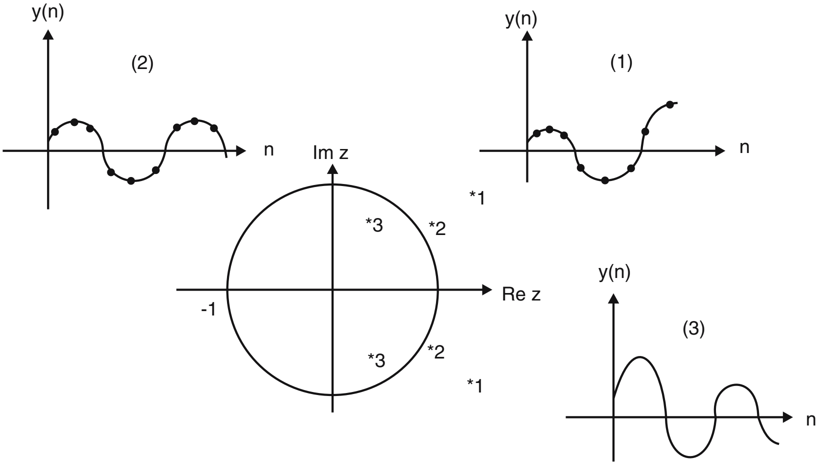


FIGURE 2.4. $(n, y(n))$ diagrams for comples roots.

(b) $\lambda_1 = \lambda_2 = \lambda$.
The general solution of (2.5.1) is given by $y(n) = (a_1 + a_2 n)\lambda^n$. Clearly, if $|\lambda| \geq 1$, the solution $y(n)$ diverges either monotonically if $\lambda \geq 1$ or by oscillating if $\lambda \leq -1$. However, if $|\lambda| < 1$, then the solution converges to zero, since $\lim_{n\to\infty} n\lambda^n = 0$ (Why?).

(c) Complex roots: $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, where $\beta \neq 0$.
As we have seen in Section 2.3, formula (2.3.12), the solution of (2.5.1) is given by $y(n) = ar^n \cos(n\theta - \omega)$, where

$$r = \sqrt{\alpha^2 + \beta^2}, \qquad \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right).$$

The solution $y(n)$ clearly oscillates, since the cosine function oscillates. However, $y(n)$ oscillates in three different ways depending on the location of the conjugate characteristic roots, as may be seen in Figure 2.4.

1. $r > 1$: Here λ_1 and $\lambda_2 = \overline{\lambda_1}$ are outside the unit circle. Hence $y(n)$ is oscillating but increasing in magnitude (unstable system).

2. $r = 1$: Here λ_1 and $\lambda_2 = \overline{\lambda_1}$ lie on the unit circle. In this case $y(n)$ is oscillating but constant in magnitude.

3. $r < 1$: Here λ_1 and $\lambda_2 = \overline{\lambda_1}$ lie inside the unit disk. The solution $y(n)$ oscillates but converges to zero as $n \to \infty$ (stable system).

Finally, we summarize the above discussion in the following theorem.

Theorem 2.35. *The following statements hold:*

(i) *All solutions of* (2.5.1) *oscillate (about zero) if and only if the characteristic equation has no positive real roots.*

(ii) *All solutions of* (2.5.1) *converge to zero (i.e., the zero solution is asymptotically stable) if and only if* $\max\{|\lambda_1|, |\lambda_2|\} < 1$.

Next we consider nonhomogeneous difference equations in which the input is constant, that is, equations of the form

$$y(n+2) + p_1 y(n+1) + p_2 y(n) = M, \tag{2.5.2}$$

where M is a nonzero constant input or forcing term. Unlike (2.5.1), the zero sequence $y(n) = 0$ for all $n \in \mathbb{Z}^+$ is not a solution of (2.5.2). Instead, we have the equilibrium point or solution $y(n) = y^*$. From (2.5.2) we have

$$y^* + p_1 y^* + p_2 y^* = M,$$

or

$$y^* = \frac{M}{1 + p_1 + p_2}. \tag{2.5.3}$$

Thus $y_p(n) = y^*$ is a particular solution of (2.5.2). Consequently, the general solution of (2.5.2) is given by

$$y(n) = y^* + y_c(n). \tag{2.5.4}$$

It is clear that $y(n) \to y^*$ if and only if $y_c(n) \to 0$ as $n \to \infty$. Furthermore, $y(n)$ oscillates[6] about y^* if and only if $y_c(n)$ oscillates about zero. These observations are summarized in the following theorem.

Theorem 2.36. *The following statements hold:*

(i) *All solutions of the nonhomogeneous equation* (2.5.2) *oscillate about the equilibrium solution y^* if and only if none of the characteristic roots of the homogeneous equation* (2.5.1) *is a positive real number.*

(ii) *All solutions of* (2.5.2) *converge to y^* as $n \to \infty$ if and only if* $\max\{|\lambda_1|, |\lambda_2|\} < 1$, *where λ_1 and λ_2 are the characteristic roots of the homogeneous equation* (2.5.1).

Theorems 2.35 and 2.36 give necessary and sufficient conditions under which a second-order difference equation is asymptotically stable. In many applications, however, one needs to have explicit criteria for stability based on the values of the coefficients p_1 and p_2 of (2.5.2) or (2.5.1). The following result provides us with such needed criteria.

[6]We say $y(n)$ oscillates about y^* if $y(n) - y^*$ alternates sign, i.e., if $y(n) > y^*$, then $y(n+1) < y^*$.

Theorem 2.37. *The conditions*

$$1 + p_1 + p_2 > 0, \qquad 1 - p_1 + p_2 > 0, \qquad 1 - p_2 > 0 \tag{2.5.5}$$

are necessary and sufficient for the equilibrium point (solution) of equations (2.5.1) *and* (2.5.2) *to be asymptotically stable (i.e., all solutions converge to* y^**).*

PROOF. Assume that the equilibrium point of (2.5.1) or (2.5.2) is asymptotically stable. In virtue of Theorems 2.35 and 2.36, the roots λ_1, λ_2 of the characteristic equation $\lambda^2 + p_1\lambda + p_2 = 0$ lie inside the unit disk, i.e., $|\lambda_1| < 1$ and $|\lambda_2| < 1$. By the quadratic formula, we have

$$\lambda_1 = \frac{-p_1 + \sqrt{p_1^2 - 4p_2}}{2} \quad \text{and} \quad \lambda_2 = \frac{-p_1 - \sqrt{p_1^2 - 4p_2}}{2}. \tag{2.5.6}$$

Then we have two cases to consider.

Case 1. λ_1, λ_2 are real roots, i.e., $p_1^2 - 4p_2 \geq 0$. From formula (2.5.6) we have

$$-2 < -p_1 + \sqrt{p_1^2 - 4p_2} < 2,$$

or

$$-2 + p_1 < \sqrt{p_1^2 - 4p_2} < 2 + p_1. \tag{2.5.7}$$

Similarly, one obtains

$$-2 + p_1 < -\sqrt{p_1^2 - 4p_2} < 2 + p_1. \tag{2.5.8}$$

Squaring the second inequality in expression (2.5.7) yields

$$1 + p_1 + p_2 > 0. \tag{2.5.9}$$

Similarly, if we square the first inequality in expression (2.5.8) we obtain

$$1 - p_1 + p_2 > 0. \tag{2.5.10}$$

Now from the second inequality of (2.5.7) and the first inequality of (2.5.8) we obtain

$$2 + p_1 > 0 \quad \text{and} \quad 2 - p_1 > 0 \quad \text{or} \quad |p_1| < 2$$

since $p_1^2 - 4p_2 \geq 0$, $p_2 \leq p_1^2/4 < 1$. This completes the proof of (2.5.5) in this case.

Case 2. λ_1 and λ_2 are complex conjugates, i.e., $p_1^2 - 4p_2 < 0$. In this case we have

$$\lambda_{1,2} = \frac{-p_1}{2} \pm \frac{i}{2}\sqrt{4p_2 - p_1^2}.$$

Moreover, since $p_1^2 < 4p_2$, it follows that $-2\sqrt{p_2} < p_1 < 2\sqrt{p_2}$. Now $|\lambda_1|^2 = \frac{p_1^2}{4} + \frac{4p_2}{4} - \frac{p_1^2}{4} = p_2$. Since $|\lambda_1| < 1$, it follows that $0 < p_2 < 1$.

Hence to show that the first two inequalities of (2.5.5) hold we need to show that the function $f(x) = 1 + x - 2\sqrt{x} > 0$ for $x \in (0,1)$. Observe that $f(0) = 1$, and $f'(x) = 1 - \frac{1}{\sqrt{x}}$. Thus $x = 1$ is a local minimum as $f(x)$ decreases for $x \in (0,1)$. Hence $f(x) > 0$ for all $x \in (0,1)$.

This completes the proof of the necessary conditions. The converse is left to the reader as Exercises 2.5, Problem 8. □

Example 2.38. Find conditions under which the solutions of the equation

$$y(n+2) - \alpha(1+\beta)y(n+1) + \alpha\beta y(n) = 1, \qquad \alpha, \beta > 0,$$

(a) converge to the equilibrium point y^*, and

(b) oscillate about y^*.

Solution Let us first find the equilibrium point y^*. Be letting $y(n) = y^*$ in the equation, we obtain

$$y^* = \frac{1}{1-\alpha}, \qquad \alpha \neq 1.$$

(a) Applying condition (2.5.5) to our equation yields

$$\alpha < 1, \qquad 1 + \alpha + 2\alpha\beta > 0, \qquad \alpha\beta < 1.$$

Clearly, the second inequality $1+\alpha+2\alpha\beta > 0$ is always satisfied, since α, β are both positive numbers.

(b) The solutions are oscillatory about y^* if either λ_1, λ_2 are negative real numbers or complex conjugates. In the first case we have

$$\alpha^2(1+\beta)^2 > 4\alpha\beta, \quad \text{or} \quad \alpha > \frac{4\beta}{(1+\beta)^2},$$

and

$$\alpha(1+\beta) < 0,$$

which is impossible. Thus if $\alpha > 4\beta/(1+\beta)^2$, we have no oscillatory solutions.

Now, λ_1 and λ_2 are complex conjugates if

$$\alpha^2(1+\beta)^2 < 4\alpha\beta \quad \text{or} \quad \alpha < \frac{4\beta}{(1+\beta)^2}.$$

Hence all solutions are oscillatory if

$$\alpha < \frac{4\beta}{(1+\beta)^2}.$$

For the treatment of the general kth-order scalar difference equations, the reader is referred to Chapter 4, on stability, and Chapter 8, on oscillation.

Exercises 2.5.

In Problems 1 through 4:

(a) Determine the stability of the equilibrium point by using Theorem 2.35 or Theorem 2.36.

(b) Determine the oscillatory behavior of the solutions of the equation.

1. $y(n+2) - 2y(n+1) + 2y(n) = 0$.
2. $y(n+2) + \frac{1}{4}y(n) = \frac{5}{4}$.
3. $y(n+2) + y(n+1) + \dfrac{1}{2}y(n) = -5$.
4. $y(n+2) - 5y(n+1) + 6y(n) = 0$.
5. Determine the stability of the equilibrium point of the equations in Problems 1 through 4 by using Theorem 2.37.
6. Show that the stability conditions (2.5.5) for the equation $y(n+2) - \alpha y(n+1) + \beta y(n) = 0$, where α, β are constants, may be written as
$$-1 - \beta < \alpha < 1 + \beta, \qquad \beta < 1.$$
7. Contemplate the equation $y(n+2) - p_1 y(n+1) - p_2 y(n) = 0$. Show that if $|p_1| + |p_2| < 1$, then all solutions of the equation converge to zero.
8. Prove that conditions (2.5.5) imply that all solutions of (2.5.2) converge to the equilibrium point y^*.
9. Determine conditions under which all solutions of the difference equation in Problem 7 oscillate.
10. Determine conditions under which all solutions of the difference equation in Problem 6 oscillate.
11. Suppose that p is a real number. Prove that every solution of the difference equation $y(n+2) - y(n+1) + py(n) = 0$ oscillates if and only if $p > \frac{1}{4}$.

*12. Prove that a necessary and sufficient condition for the asymptotic stability of the zero solution of the equation
$$y(n+2) + p_1 y(n+1) + p_2 y(n) = 0$$
is
$$|p_1| < 1 + p_2 < 2.$$

13. Determine the limiting behavior of solutions of the equation

$$y(n+2) = \alpha c + \alpha\beta(y(n+1) - y(n))$$

if:

(i) $\alpha\beta = 1$,

(ii) $\alpha\beta = 2$,

(iii) $\alpha\beta = \frac{1}{2}$,

provided that α, β, and c are positive constants.

14. If $p_1 > 0$ and $p_2 > 0$, show that all solutions of the equation

$$y(n+2) + p_1 y(n+1) + p_2 y(n) = 0$$

are oscillatory.

15. Determine the limiting behavior of solutions of the equation

$$y(n+2) - \frac{\beta}{\alpha} y(n+1) + \frac{\beta}{\alpha} y(n) = 0,$$

where α and β are constants, if:

(i) $\beta > 4\alpha$,

(ii) $\beta < 4\alpha$.

2.6 Nonlinear Equations Transformable to Linear Equations

In general, most nonlinear difference equations cannot be solved explicitly. However, a few types of nonlinear equations can be solved, usually by transforming them into linear equations. In this section we discuss some tricks of the trade.

Type I. Equations of Riccati type:

$$x(n+1)x(n) + p(n)x(n+1) + q(n)x(n) = 0. \tag{2.6.1}$$

To solve the Riccati equation, we let

$$z(n) = \frac{1}{x(n)}$$

in (2.6.1) to give us

$$q(n)z(n+1) + p(n)z(n) + 1 = 0. \tag{2.6.2}$$

The nonhomogeneous equation requires a different transformation

$$y(n+1)y(n) + p(n)y(n+1) + q(n)y(n) = g(n). \tag{2.6.3}$$

If we let $y(n) = (z(n+1)/z(n)) - p(n)$ in (2.6.3) we obtain

$$z(n+2) + (q(n) - p(n+1))z(n+1) - (g(n) + p(n)q(n))z(n) = 0.$$

Example 2.39. The Pielou Logistic Equation

The most popular continuous model of the growth of a population is the well-known Verhulst–Pearl equation given by

$$x'(t) = x(t)[a - bx(t)], \qquad a, b > 0, \tag{2.6.4}$$

where $x(t)$ is the size of the population at time t; a is the rate of the growth of the population if the resources were unlimited and the individuals did not affect one another, and $-bx^2(t)$ represents the negative effect on the growth due to crowdedness and limited resources. The solution of (2.6.4) is given by

$$x(t) = \frac{a/b}{1 + (e^{-at}/cb)}.$$

Now,

$$\begin{aligned} x(t+1) &= \frac{a/b}{1 + \left(e^{-a(t+1)}/cb\right)} \\ &= \frac{e^a(a/b)}{1 + (e^{-at}/cb) + (e^a - 1)}. \end{aligned}$$

Dividing by $[1 + (e^{-at}/cb)]$, we obtain

$$x(t+1) = \frac{e^a x(t)}{\left[1 + \frac{b}{a}(e^a - 1)x(t)\right]},$$

or

$$x(n+1) = \frac{\alpha x(n)}{[1 + \beta x(n)]}, \tag{2.6.5}$$

where $\alpha = e^a$ and $\beta = \frac{b}{a}(e^a - 1)$.

This equation is titled the *Pielou logistic equation.*

Equation (2.6.5) is of Riccati type and may be solved by letting $x(n) = 1/z(n)$. This gives us the equation

$$z(n+1) = \frac{1}{\alpha} z(n) + \frac{\beta}{\alpha},$$

whose solution is given by

$$z(n) = \begin{cases} \left[c - \dfrac{\beta}{\alpha - 1}\right] \alpha^{-n} + (\beta/(\alpha - 1)) & \text{if } \alpha \neq 1, \\ c + \beta n & \text{if } \alpha = 1. \end{cases}$$

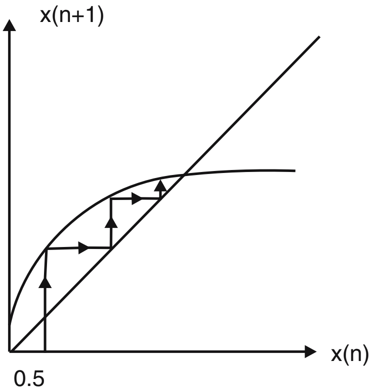


FIGURE 2.5. Asymptotically stable equilibrium points.

Thus

$$x(n) = \begin{cases} \alpha^n(\alpha-1)/[\beta\alpha^n + c(\alpha-1) - \beta] & \text{if } \alpha \neq 1, \\ \dfrac{1}{c+\beta n} & \text{if } \alpha = 1. \end{cases}$$

Hence

$$\lim_{n\to\infty} x(n) = \begin{cases} (\alpha-1)/\beta & \text{if } \alpha \neq 1, \\ 0 & \text{if } \alpha = 1. \end{cases}$$

This conclusion shows that the equilibrium point $(\alpha - 1)/\beta$ is globally asymptotically stable if $\alpha \neq 1$. Figure 2.5 illustrates this for $\alpha = 3$, $\beta = 1$, and $x(0) = 0.5$.

Type II. Equations of general Riccati type:

$$x(n+1) = \frac{a(n)x(n) + b(n)}{c(n)x(n) + d(n)} \tag{2.6.6}$$

such that $c(n) \neq 0, a(n)d(n) - b(n)c(n) \neq 0$ for all $n \geq 0$.

To solve this equation we let

$$c(n)x(n) + d(n) = \frac{y(n+1)}{y(n)}. \tag{2.6.7}$$

Then by substituting

$$x(n) = \frac{y(n+1)}{c(n)y(n)} - \frac{d(n)}{c(n)}$$

into (2.6.6) we obtain

$$\frac{y(n+2)}{c(n+1)y(n+1)} - \frac{d(n+1)}{c(n+1)} = \frac{a(n)\left[\frac{y(n+1)}{c(n)y(n)} - \frac{d(n)}{c(n)}\right] + b(n)}{\frac{y(n+1)}{y(n)}}.$$

This equation simplifies to

$$y(n+2) + p_1(n)y(n+1) + p_2(n)y(n) = 0,$$
$$y(0) = 1, \quad y(1) = c(0)x(0) + d(0), \tag{2.6.8}$$

where

$$p_1(n) = -\frac{c(n)d(n+1) + a(n)c(n+1)}{c(n)},$$
$$p_2(n) = (a(n)d(n) - b(n)c(n))\frac{c(n+1)}{c(n)}.$$

Example 2.40. Solve the difference equation

$$x(n+1) = \frac{2x(n)+3}{3x(n)+2}.$$

Solution Here $a = 2, b = 3, c = 3$, and $d = 2$. Hence $ad - bc \neq 0$. Using the transformation

$$3x(n) + 2 = \frac{y(n+1)}{y(n)}, \tag{2.6.9}$$

we obtain, as in (2.6.8),

$$y(n+2) - 4y(n+1) - 5y(n) = 0, \qquad y(0) = 1, \qquad y(1) = 3x(0) + 2,$$

with characteristic roots $\lambda_1 = 5, \lambda_2 = -1$.

Hence

$$y(n) = c_1 5^n + c_2(-1)^n. \tag{2.6.10}$$

From formula (2.6.9) we have

$$x(n) = \frac{1}{3}\frac{y(n+1)}{y(n)} - \frac{2}{3} = \frac{1}{3}\frac{c_1 5^{n+1} + c_2(-1)^{n+1}}{c_1 5^n + c_2(-1)^n} - \frac{2}{3}$$
$$= \frac{(c_1 5^n - c_2(-1)^n)}{(c_1 5^n + c_2(-1)^n)} = \frac{5^n - c(-1)^n}{5^n + c(-1)^n},$$

where

$$c = \frac{c_1}{c_2}.$$

Type III. Homogeneous difference equations of the type

$$f\left(\frac{x(n+1)}{x(n)}, n\right) = 0.$$

Use the transformation $z(n) = \frac{x(n+1)}{x(n)}$ to convert such an equation to a linear equation in $z(n)$, thus allowing it to be solved.

Example 2.41. Solve the difference equation

$$x^2(n+1) - 3x(n+1)x(n) + 2x^2(n) = 0. \tag{2.6.11}$$

Solution Dividing by $x^2(n)$, equation (2.6.11) becomes

$$\left[\frac{x(n+1)}{x(n)}\right]^2 - 3\left[\frac{x(n+1)}{x(n)}\right] + 2 = 0, \tag{2.6.12}$$

which is of Type III.

Letting $z(n) = \frac{x(n+1)}{x(n)}$ in (2.6.12) creates

$$z^2(n) - 3z(n) + 2 = 0.$$

We can factor this down to

$$[z(n) - 2][z(n) - 1] = 0,$$

and thus either $z(n) = 2$ or $z(n) = 1$.

This leads to

$$x(n+1) = 2x(n) \quad \text{or} \quad x(n+1) = x(n).$$

Starting with $x(0) = x_0$, there are infinitely many solutions $x(n)$ of (2.6.11) of the form

$$x_0, \ldots, x_0; 2x_0, \ldots, 2x_0; 2^2x_0, \ldots, 2^2x_0; \ldots .$$[7]

Type IV. Consider the difference equation of the form

$$(y(n+k))^{r_1} (y(n+k-1))^{r_2} \cdots (y(n))^{r_{k+1}} = g(n). \tag{2.6.13}$$

Let $z(n) = \ln y(n)$, and rearrange to obtain

$$r_1 z(n+k) + r_2 z(n+k-1) + \cdots + r_{k+1} z(n) = \ln g(n). \tag{2.6.14}$$

Example 2.42. Solve the difference equation

$$x(n+2) = \frac{x^2(n+1)}{x^2(n)}. \tag{2.6.15}$$

Solution Let $z(n) = \ln x(n)$ in (2.6.15). Then as in (2.6.12) we obtain

$$z(n+2) - 2z(n+1) + 2z(n) = 0.$$

The characteristic roots are $\lambda_1 = 1 + i, \lambda_2 = 1 - i$.

Thus,

$$z(n) = (2)^{n/2}\left[c_1 \cos\left(\frac{n\pi}{4}\right) + c_2 \sin\left(\frac{n\pi}{4}\right)\right].$$

[7]This solution was given by Sebastian Pancratz of the Technical University of Munich.

Therefore,

$$x(n) = \exp\left[(2)^{n/2}\left\{c_1 \cos\left(\frac{n\pi}{4}\right) + c_2 \sin\left(\frac{n\pi}{4}\right)\right\}\right].$$

Exercises 2.6

1. Find the general solution of the difference equation
$$y^2(n+1) - 2y(n+1)y(n) - 3y^2(n) = 0.$$
2. Solve the difference equation
$$y^2(n+1) - (2+n)y(n+1)y(n) + 2ny^2(n) = 0.$$
3. Solve $y(n+1)y(n) - y(n+1) + y(n) = 0$.
4. Solve $y(n+1)y(n) - \frac{2}{3}y(n+1) + \frac{1}{6}y(n) = \frac{5}{18}$.
5. Solve $y(n+1) = 5 - \dfrac{6}{y(n)}$.
6. Solve $x(n+1) = \dfrac{x(n)+a}{x(n)+1}, 1 \neq a > 0$.
7. Solve $x(n+1) = x^2(n)$.
8. Solve the logistic difference equation
$$x(n+1) = 2x(n)(1 - x(n)).$$
9. Solve the logistic equation
$$x(n+1) = 4x(n)[1 - x(n)].$$
10. Solve $x(n+1) = \dfrac{1}{2}\left(x(n) - \dfrac{a}{x(n)}\right), a > 0$.
11. Solve $y(n+2) = y^3(n+1)/y^2(n)$.
12. Solve $x(n+1) = \dfrac{2x(n)+4}{x(n)-1}$.
13. Solve $y(n+1) = \dfrac{2 - y^2(n)}{2(1 - y(n))}$.
14. Solve $x(n+1) = \dfrac{2x(n)}{x(n)+3}$.
15. Solve $y(n+1) = 2y(n)\sqrt{1 - y^2(n)}$.
16. The "regular falsi" method for finding the roots of $f(x) = 0$ is given by
$$x(n+1) = \frac{x(n-1)f(x(n)) - x(n)f(x(n-1))}{f(x(n)) - f(x(n-1))}.$$

(a) Show that for $f(x) = x^2$, this difference equation becomes

$$x(n+1) = \frac{x(n-1)x(n)}{x(n-1)+x(n)}.$$

(b) Let $x(1) = 1, x(2) = 1$ for the equation in part (a). Show that the solution of the equation is $x(n) = 1/F(n)$, where $F(n)$ is the nth Fibonacci number.

2.7 Applications

2.7.1 Propagation of Annual Plants

The material of this section comes from Edelstein–Keshet [37] of plant propagation. Our objective here is to develop a mathematical model that describes the number of plants in any desired generation. It is known that plants produce seeds at the end of their growth season (say August), after which they die. Furthermore, only a fraction of these seeds survive the winter, and those that survive germinate at the beginning of the season (say May), giving rise to a new generation of plants.

Let

$$\begin{aligned}
\gamma &= \text{number of seeds produced per plant in August,}\\
\alpha &= \text{fraction of one-year-old seeds that germinate in May,}\\
\beta &= \text{fraction of two-year-old seeds that germinate in May,}\\
\sigma &= \text{fraction of seeds that survive a given winter.}
\end{aligned}$$

If $p(n)$ denotes the number of plants in generation n, then

$$p(n) = \begin{pmatrix}\text{plants from}\\ \text{one-year-old seeds}\end{pmatrix} + \begin{pmatrix}\text{plants from}\\ \text{two-year-old seeds}\end{pmatrix},$$

$$p(n) = \alpha s_1(n) + \beta s_2(n), \tag{2.7.1}$$

where $s_1(n)$ (respectively, $s_2(n)$) is the number of one-year-old (two-year-old) seeds in April (before germination). Observe that the number of seeds left after germination may be written as

$$\text{seeds left} = \begin{pmatrix}\text{fraction}\\ \text{not germinated}\end{pmatrix} \times \begin{pmatrix}\text{original number}\\ \text{of seeds in April}\end{pmatrix}.$$

This gives rise to two equations:

$$\tilde{s}_1(n) = (1-\alpha)s_1(n), \tag{2.7.2}$$

$$\tilde{s}_2(n) = (1-\beta)s_2(n), \tag{2.7.3}$$

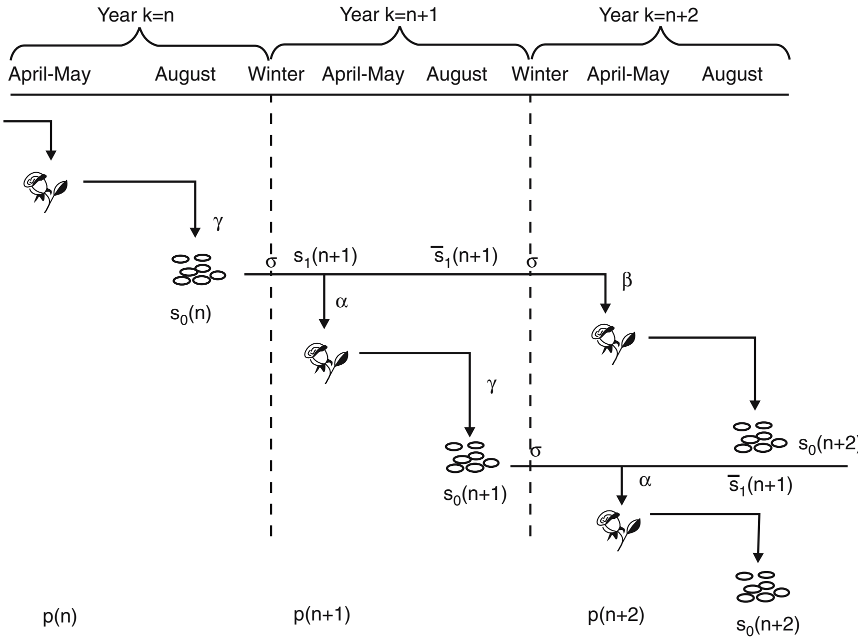


FIGURE 2.6. Propogation of annual plants.

where $\tilde{s}_1(n)$ (respectively, $\tilde{s}_2(n)$) is the number of one-year (two-year-old) seeds left in May after some have germinated. New seeds $s_0(n)$ (0-year-old) are produced in August (Figure 2.6) at the rate of γ per plant,

$$s_0(n) = \gamma p(n). \tag{2.7.4}$$

After winter, seeds $s_0(n)$ that were new in generation n will be one year old in the next generation $n+1$, and a fraction $\sigma s_0(n)$ of them will survive. Hence

$$s_1(n+1) = \sigma s_0(n),$$

or, by using formula (2.7.4), we have

$$s_1(n+1) = \sigma \gamma p(n). \tag{2.7.5}$$

Similarly,

$$s_2(n+1) = \sigma \tilde{s}_1(n),$$

which yields, by formula (2.7.2),

$$\begin{aligned} s_2(n+1) &= \sigma(1-\alpha)s_1(n), \\ s_2(n+1) &= \sigma^2\gamma(1-\alpha)p(n-1). \end{aligned} \tag{2.7.6}$$

Substituting for $s_1(n+1), s_2(n+1)$ in expressions (2.7.5) and (2.7.6) into formula (2.7.1) gives

$$p(n+1) = \alpha\gamma\sigma p(n) + \beta\gamma\sigma^2(1-\alpha)p(n-1),$$

or

$$p(n+2) = \alpha\gamma\sigma p(n+1) + \beta\gamma\sigma^2(1-\alpha)p(n). \tag{2.7.7}$$

The characteristic equation (2.7.7) is given by

$$\lambda^2 - \alpha\gamma\sigma\lambda - \beta\gamma\sigma^2(1-\alpha) = 0$$

with characteristic roots

$$\lambda_1 = \frac{\alpha\gamma\sigma}{2}\left[1 + \sqrt{1 + \frac{4\beta}{\gamma\alpha^2}(1-\alpha)}\,\right],$$

$$\lambda_2 = \frac{\alpha\gamma\sigma}{2}\left[1 - \sqrt{1 + \frac{4\beta}{\gamma\alpha^2}(1-\alpha)}\,\right].$$

Observe that λ_1 and λ_2 are real roots, since $1-\alpha > 0$. Furthermore, $\lambda_1 > 0$ and $\lambda_2 < 0$. To ensure propagation (i.e., $p(n)$ increases indefinitely as $n \to \infty$) we need to have $\lambda_1 > 1$. We are not going to do the same with λ_2, since it is negative and leads to undesired fluctuation (oscillation) in the size of the plant population. Hence

$$\frac{\alpha\gamma\sigma}{2}\left[1 + \sqrt{1 + \frac{4\beta}{\gamma\alpha^2}(1-\alpha)}\,\right] > 1,$$

or

$$\frac{\alpha\gamma\sigma}{2}\sqrt{1 + \frac{4\beta(1-\alpha)}{\gamma\alpha^2}} > 1 - \frac{\alpha\gamma\sigma}{2}.$$

Squaring both sides and simplifying yields

$$\gamma > \frac{1}{\alpha\sigma + \beta\sigma^2(1-\alpha)}. \tag{2.7.8}$$

If $\beta = 0$, that is, if no two-year-old seeds germinate in May, then condition (2.7.8) becomes

$$\gamma > \frac{1}{\alpha\sigma}. \tag{2.7.9}$$

Condition (2.7.9) says that plant propagation occurs if the product of the fraction of seeds produced per plant in August, the fraction of one-year-old seeds that germinate in May, and the fraction of seeds that survive a given winter exceeds 1.

2.7.2 Gambler's Ruin

A gambler plays a sequence of games against an adversary in which the probability that the gambler wins \$1.00 in any given game is a known value q, and the probability of his losing \$1.00 is $1-q$, where $0 \leq q \leq 1$. He quits gambling if he either loses all his money or reaches his goal of acquiring N dollars. If the gambler runs out of money first, we say that the gambler has been ruined. Let $p(n)$ denote the probability that the gambler will be ruined if he possesses n dollars. He may be ruined in two ways. First, winning the next game; the probability of this event is q; then his fortune will be $n+1$, and the probability of being ruined will become $p(n+1)$. Second, losing the next game; the probability of this event is $1-q$, and the probability of being ruined is $p(n-1)$. Hence applying the theorem of total probabilities, we have

$$p(n) = qp(n+1) + (1-q)p(n-1).$$

Replacing n by $n+1$, we get

$$p(n+2) - \frac{1}{q}p(n+1) + \frac{(1-q)}{q}p(n) = 0, \quad n = 0, 1, \ldots, N, \tag{2.7.10}$$

with $p(0) = 1$ and $p(N) = 0$. The characteristic equation is given by

$$\lambda^2 - \frac{1}{q}\lambda + \frac{1-q}{q} = 0,$$

and the characteristic roots are given by

$$\lambda_1 = \frac{1}{2q} + \frac{1-2q}{2q} = \frac{1-q}{q},$$
$$\lambda_2 = \frac{1}{2q} - \frac{1-2q}{2q} = 1.$$

Hence the general solution may be written as

$$p(n) = c_1 + c_2\left(\frac{1-q}{q}\right)^n, \qquad \text{if } q \neq \frac{1}{2}.$$

Now using the initial conditions $p(0) = 1, P(N) = 0$ we obtain

$$c_1 + c_2 = 1, \qquad c_1 + c_2\left(\frac{1-q}{q}\right)^N = 0,$$

which gives

$$c_1 = \frac{-\left(\frac{1-q}{q}\right)^N}{1-\left(\frac{1-q}{q}\right)^N}, \qquad c_2 = \frac{1}{1-\left(\frac{1-q}{q}\right)^N}.$$

Thus

$$p(n) = \frac{\left(\frac{1-q}{q}\right)^n - \left(\frac{1-q}{q}\right)^N}{1 - \left(\frac{1-q}{q}\right)^N}. \tag{2.7.11}$$

The special case $q = \frac{1}{2}$ must be treated separately, since in this case we have repeated roots $\lambda_1 = \lambda_2 = 1$. This is certainly the case when we have a fair game. The general solution in this case may be given by

$$p(n) = a_1 + a_2 n,$$

which with the initial conditions yields

$$p(n) = 1 - \frac{n}{N} = \frac{N-n}{N}. \tag{2.7.12}$$

For example, suppose you start with \$4, the probability that you win a dollar is 0.3, and you will quit if you run out of money or have a total of \$10. Then $n = 4$, $q = 0.3$, and $N = 10$, and the probability of being ruined is given by

$$p(4) = \frac{\left(\frac{7}{3}\right)^4 - \left(\frac{7}{3}\right)^{10}}{1 - \left(\frac{7}{3}\right)^{10}} = 0.994.$$

On the other hand, if $q = 0.5$, $N = \$100.00$, and $n = 20$, then from formula (2.7.12) we have

$$p(20) = 1 - \frac{20}{100} = 0.8.$$

Observe that if $q \leq 0.5$ and $N \to \infty$, $p(n)$ tends to 1 in both formulas (2.7.11) and (2.7.12), and the gambler's ruin is certain.

The probability that the gambler wins is given by

$$\tilde{p}(n) = 1 - p(n) = \begin{cases} \dfrac{1 - \left(\frac{1-q}{q}\right)^n}{1 - \left(\frac{1-q}{q}\right)^N}, & \text{if } q \neq 0.5, \\[2ex] \dfrac{n}{N}, & \text{if } q = 0.5. \end{cases} \tag{2.7.13}$$

2.7.3 National Income

In a capitalist country the national income $Y(n)$ in a given period n may be written as

$$Y(n) = C(n) + I(n) + G(n), \tag{2.7.14}$$

where

$C(n) =$ consumer expenditure for purchase of consumer goods,
$I(n) =$ induced private investment for buying capital equipment, and
$G(n) =$ government expenditure,

where n is usually measured in years.

We now make some assumptions that are widely accepted by economists (see, for example, Samuelson [129]).

(a) Consumer expenditure $C(n)$ is proportional to the national income $Y(n-1)$ in the preceding year $n-1$, that is,

$$C(n) = \alpha Y(n-1), \tag{2.7.15}$$

where $\alpha > 0$ is commonly called the *marginal propensity to consume.*

(b) Induced private investment $I(n)$ is proportional to the increase in consumption $C(n) - C(n-1)$, that is,

$$I(n) = \beta[C(n) - C(n-1)], \tag{2.7.16}$$

where $\beta > 0$ is called the *relation.*

(c) Finally, the government expenditure $G(n)$ is constant over the years, and we may choose our units such that

$$G(n) = 1. \tag{2.7.17}$$

Employing formulas (2.7.15), (2.7.16), and (2.7.17) in formula (2.7.14) produces the second-order difference equation

$$Y(n+2) - \alpha(1+\beta)Y(n+1) + \alpha\beta Y(n) = 1, \qquad n \in \mathbb{Z}^+. \tag{2.7.18}$$

Observe that this is the same equation we have already studied, in detail, in Example 2.38. As we have seen there, the equilibrium state of the national income $Y^* = 1/(1-\alpha)$ is asymptotically stable (or just stable in the theory of economics) if and only if the following conditions hold:

$$\alpha < 1, \qquad 1 + \alpha + 2\alpha\beta > 0, \qquad \alpha\beta < 1. \tag{2.7.19}$$

Furthermore, the national income $Y(n)$ fluctuates (oscillates) around the equilibrium state Y^* if and only if

$$\alpha < \frac{4\beta}{(1+\beta)^2}. \tag{2.7.20}$$

Now consider a concrete example where $\alpha = \frac{1}{2}, \beta = 1$. Then $Y^* = 2$, i.e., $Y^* =$ twice the government expenditure. Then clearly, conditions (2.7.19) and (2.7.12) are satisfied. Hence the national income $Y(n)$ always converges in an oscillatory fashion to $Y^* = 2$, regardless of what the initial national income $Y(0)$ and $Y(1)$ are. (See Figure 2.7.)

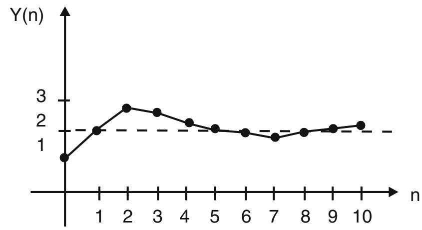


FIGURE 2.7. Solution of $Y(n+2) - Y(n+1) + Y(n) = 1, Y(0) = 1, Y(1) = 2$.

The actual solution may be given by

$$Y(n) = A\left(\frac{1}{\sqrt{2}}\right)^n \cos\left(\frac{n\pi}{4} - \omega\right) + 2.$$

Figure 2.7 depicts the solution $Y(n)$ if $Y(0) = 1$ and $Y(1) = 2$. Here we find that $A = -\sqrt{2}$ and $\omega = \pi/4$ and, consequently, the solution is

$$Y(n) = -\left(\frac{1}{\sqrt{2}}\right)^{n-1} \cos\left[\frac{(n+1)}{4}\pi\right] + 2.$$

Finally, Figure 2.8 depicts the parameter diagram $(\beta - \alpha)$, which shows regions of stability and regions of instability.

2.7.4 The Transmission of Information

Suppose that a signaling system has two signals s_1 and s_2 such as dots and dashes in telegraphy. Messages are transmitted by first encoding them into a string, or sequence, of these two signals. Suppose that s_1 requires exactly n_1 units of time, and s_2 exactly n_2 units of time, to be transmitted. Let $M(n)$ be the number of possible message sequences of duration n. Now, a signal of duration time n either ends with an s_1 signal or with an s_2 signal.

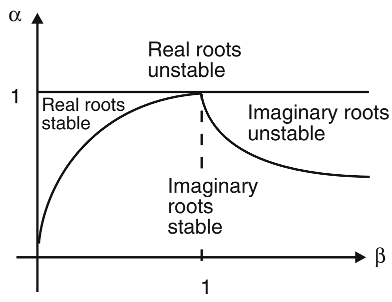


FIGURE 2.8. Parametric diagram $(\beta - \alpha)$.

FIGURE 2.9. Two signals, one ends with s_1 and the other with s_2.

If the message ends with s_1, the last signal must start at $n - n_1$ (since s_1 takes n_1 units of time). Hence there are $M(n - n_1)$ possible messages to which the last s_1 may be appended. Hence there are $M(n - n_1)$ messages of duration n that end with s_1. By a similar argument, one may conclude that there are $M(n - n_2)$ messages of duration n that end with s_2. (See Figure 2.9.) Consequently, the total number of messages $x(n)$ of duration n may be given by

$$M(n) = M(n - n_1) + M(n - n_2).$$

If $n_1 \geq n_2$, then the above equation may be written in the familiar form of an n_1th-order equation

$$M(n + n_1) - M(n + n_1 - n_2) - M(n) = 0. \tag{2.7.21}$$

On the other hand, if $n_1 \leq n_2$, then we obtain the n_2th-order equation

$$M(n + n_2) - M(n + n_2 - n_1) - M(n) = 0. \tag{2.7.22}$$

An interesting special case is that in which $n_1 = 1$ and $n_2 = 2$. In this case we have

$$M(n + 2) - M(n + 1) - M(n) = 0,$$

or

$$M(n + 2) = M(n + 1) + M(n),$$

which is nothing but our Fibonacci sequence $\{0, 1, 1, 2, 3, 5, 8, \ldots\}$, which we encountered in Example 2.27. The general solution (see formula (2.3.14)) is given by

$$M(n) = a_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + a_2 \left(\frac{1-\sqrt{5}}{2}\right)^n, \qquad n = 0, 1, 2, \ldots. \tag{2.7.23}$$

To find a_1 and a_2 we need to specify $M(0)$ and $M(1)$. Here a sensible assumption is to let $M(0) = 0$ and $M(1) = 1$. Using these initial data in (2.7.23) yields

$$a_1 = \frac{1}{\sqrt{5}}, \qquad a_2 = -\frac{1}{\sqrt{5}},$$

and the solution of our problem now becomes

$$M(n) = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{2.7.24}$$

In information theory, the capacity C of the channel is defined as

$$C = \lim_{n\to\infty} \frac{\log_2 M(n)}{n}, \tag{2.7.25}$$

where $\log_2$ denotes the logarithm base 2.

From (2.7.24) we have

$$C = \lim_{n\to\infty} \frac{\log_2 \frac{1}{\sqrt{5}}}{n} + \lim_{n\to\infty} \frac{1}{n} \log_2 \left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^n \right]. \tag{2.7.26}$$

Since $\left(\frac{1-\sqrt{5}}{2}\right) \approx 0.6 < 1$, it follows that $\left(\frac{1-\sqrt{5}}{2}\right)^n \to 0$ as $n \to \infty$.

Observe also that the first term on the right-hand side of (2.7.26) goes to zero as $n \to \infty$.

Thus

$$C = \lim_{n\to\infty} \frac{1}{n} \log_2 \left(\frac{1+\sqrt{5}}{2}\right)^n,$$

$$C = \log_2 \left(\frac{1+\sqrt{5}}{2}\right) \approx 0.7. \tag{2.7.27}$$

Exercises 2.7

1. The model for annual plants was given by (2.7.7) in terms of the plant population $p(n)$.
   (a) Write the model in terms of $s_1(n)$.
   (b) Let $\alpha = \beta = 0.01$ and $\sigma = 1$. How big should γ be to ensure that the plant population increases in size?
2. An alternative formulation for the annual plant model is that in which we define the beginning of a generation as the time when seeds are produced. Figure 2.10 shows the new method.

   Write the difference equation in $p(n)$ that represents this model. Then find conditions on γ under which plant propagation occurs.
3. A planted seed produces a flower with one seed at the end of the first year and a flower with two seeds at the end of two years and each year thereafter. Suppose that each seed is planted as soon as it is produced.
   (a) Write the difference equation that describes the number of flowers $F(n)$ at the end of the nth year.
   (b) Compute the number of flowers at the end of 3, 4, and 5 years.
4. Suppose that the probability of winning any particular bet is 0.49. If you start with \$50 and will quit when you have \$100, what is the probability of ruin (i.e., losing all your money):

(i) if you make \$1 bets?

(ii) if you make \$10 bets?

(iii) if you make \$50 bets?

5. John has m chips and Robert has $(N - m)$ chips. Suppose that John has a probability p of winning each game, where one chip is bet on in each play. If $G(m)$ is the expected value of the number of games that will be played before either John or Robert is ruined:

   (a) Show that $G(m)$ satisfies the second-order equation

   $$G(m + 2) + pG(m + 1) + (1 - p)G(m) = 0. \tag{2.7.28}$$

   (b) What are the values of $G(0)$ and $G(N)$?

   (c) Solve the difference equation (2.7.28) with the boundary conditions in part (b).

6. Suppose that in a game we have the following situation: On each play, the probability that you will win \$2 is 0.1, the probability that you will win \$1 is 0.3, and the probability that you will lose \$1 is 0.6. Suppose you quit when either you are broke or when you have at least N dollars. Write a third-order difference equation that describes the probability $p(n)$ of eventually going broke if you have n dollars. Then find the solution of the equation.

7. Suppose that Becky plays a roulette wheel that has 37 divisions: 18 are red, 18 are black, and one is green. Becky can bet on either the red or black, and she wins a sum equal to her bet if the outcome is a division of that color; otherwise, she loses the bet. If the bank has one

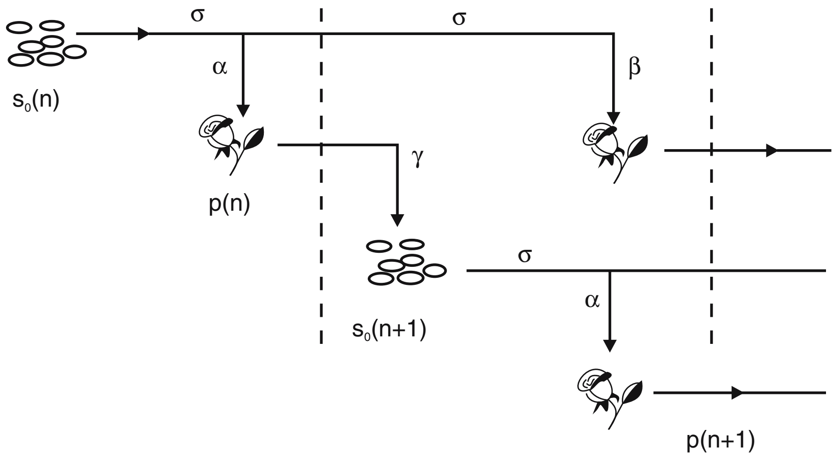


FIGURE 2.10. Annual plant model.

million dollars and she has \$5000, what is the probability that Becky can break the bank, assuming that she bets \$100 on either red or black for each spin of the wheel?

8. In the national income model (2.7.14), assume that the government expenditure $G(n)$ is proportional to the national income $Y(n-2)$ two periods past, i.e., $G(n) = \gamma Y(n-2)$, $0 < \gamma < 1$. Derive the difference equation for the national income $Y(n)$. Find the conditions for stability and oscillations of solutions.

9. Determine the behavior (stability, oscillations) of solutions of (2.7.18) for the cases:

   (a) $\alpha = \dfrac{4\beta}{(1+\beta)^2}$.

   (b) $\alpha > \dfrac{4\beta}{(1+\beta)^2}$.

10. Modify the national income model such that instead of the government having fixed expenditures, it increases its expenditures by 5% each time period, that is, $G(n) = (1.05)^n$.

    (a) Write down the second-order difference equation that describes this model.

    (b) Find the equilibrium value.

    (c) If $\alpha = 0.5, \beta = 1$, find the general solution of the equation.

11. Suppose that in the national income we make the following assumptions:

    (i) $Y(n) = C(n) + I(n)$, i.e., there is no government expenditure.

    (ii) $C(n) = a_1Y(n-1) + a_2Y(n-2) + K$, i.e., consumption in any period is a linear combination of the incomes of the two preceding periods, where a_1, a_2, and K are constants.

    (iii) $I(n+1) = I(n) + h$, i.e., investment increases by a fixed amount $h > 0$ each period.

    (a) Write down a third-order difference equation that models the national income $Y(n)$.

    (b) Find the general solution if $a_1 = \dfrac{1}{2}, a_2 = \dfrac{1}{4}$.

    (c) Show that $Y(n)$ is asymptotic to the equilibrium $Y^* = \alpha + \beta n$.

12. (Inventory Analysis). Let $S(n)$ be the number of units of consumer goods produced for sale in period n, and let $T(n)$ be the number of units of consumer goods produced for inventories in period n. Assume that there is a constant noninduced net investment V_0 in each period.

Then the total income $Y(n)$ produced in time n is given by $Y(n) = T(n) + S(n) + V_0$.

(a) Develop a difference equation that models the total income $Y(n)$, under the assumptions:

(i) $S(n) = \beta Y(n-1)$,

(ii) $T(n) = \beta Y(n-1) - \beta Y(n-2)$.

(b) Obtain conditions under which:

(i) solutions converge to the equilibrium,

(ii) solutions are oscillatory.

(c) Interpret your results in part (b).

13. Let $I(n)$ denote the level of inventories at the close of period n.

(a) Show that $I(n) = I(n-1) + S(n) + T(n) - \beta Y(n)$ where $S(n), T(n), Y(n)$ are as in Problem 12.

(b) Assuming that $S(n) = 0$ (passive inventory adjustment), show that

$$I(n) - I(n-1) = (1-\beta)Y(n) - V_0$$

where V_0 is as in Problem 12.

(c) Suppose as in part (b) that $s(n) = 0$. Show that

$$I(n+2) - (\beta+1)I(n+1) + \beta I(n) = 0.$$

(d) With $\beta \neq 1$, show that

$$I(n) = \left(I(0) - \frac{c}{1-\beta}\right)\beta^n + \frac{c}{1-\beta},$$

where $(E-\beta)I(n) = c$.

14. Consider (2.7.21) with $n_1 = n_2 = 2$ (i.e., both signals s_1 and s_2 take two units of time for transmission).

(a) Solve the obtained difference equation with the initial conditions $M(2) = M(3) = 2$.

(b) Find the channel capacity c.

15. Consider (2.7.21) with $n_1 = n_2 = 1$ (i.e., both signals take one unit of time for transmission).

(a) Solve the obtained difference equation.

(b) Find the channel capacity c.

16. (Euler's method for solving a second-order differential equation.) Recall from Section 1.4.1 that one may approximate $x'(t)$ by $(x(n+1) - x(n))/h$, where h is the step size of the approximation and $x(n) = x(t_0 + nh)$.

    (a) Show that $x''(t)$ may be approximated by

    $$\frac{x(n+2) - 2x(n+1) + x(n)}{h^2}.$$

    (b) Write down the corresponding difference equation of the differential equation

    $$x''(t) = f(x(t), x'(t)).$$

17. Use Euler's method described in Problem 16 to write the corresponding difference equation of

    $$x''(t) - 4x(t) = 0, \quad x(0) = 0, \quad x'(0) = 1.$$

    Solve both differential and difference equations and compare the results.

18. (The Midpoint Method). The midpoint method stipulates that one may approximate $x'(t)$ by $(x(n+1) - x(n-1))/h$, where h is the step size of the approximation and $t = t_0 + nh$.

    (a) Use the method to write the corresponding difference equation of the differential equation $x'(t) = g(t, x(t))$.

    (b) Use the method to write the corresponding difference equation of $x'(t) = 0.7x^2 + 0.7, x(0) = 1, t \in [0, 1]$. Then solve the obtained difference equation.

    (c) Compare your findings in part (b) with the results in Section 1.4.1. Determine which of the two methods, Euler or midpoint, is more accurate.