2
Recurrence

It often happens that, in studying a sequence of numbers a_n, a connection between a_n and a_{n-1}, or between a_n and several of the previous $a_i, i < n$, is obtained. This connection is called a recurrence relation; it is the aim of this chapter to illustrate how such recurrences arise and how they may be solved.

2.1 Some Examples

Example 2.1 (The towers of Hanoi)

We begin with a problem made famous by the nineteenth century French mathematician E. Lucas. Consider n discs, all of different sizes, with holes at their centres (like old gramophone records), and three vertical poles onto which the discs can be slipped. Initially all the discs are on one of the poles, in order of size, with the largest at the bottom, forming a tower. It is required to move the discs, one at a time, finishing up with the n discs similarly arranged on one of the other poles. There is the important requirement that at no stage may any disc be placed on top of a smaller disc. What is the minimum number of moves required?

Let a_n denote the smallest number of moves required to move the n discs. Then clearly $a_1 = 1$. Also, $a_2 = 3$: move the top disc to one pole, the bottom to the other, and then place the smaller on top of the larger. What about a_n? It should be clear that, to be able to move the bottom disc, there has to be an empty pole to move it to, and so all the other $n-1$ discs must have been moved to the third pole. To get to this stage, a_{n-1} moves are needed. The largest disc is then moved to the free pole, and then another a_{n-1} moves can position the other $n-1$ discs on top of it. So

$$a_n = 2a_{n-1} + 1.$$

This **recurrence relation**, along with the **initial condition** $a_1 = 1$, enables us to find a_n. We have $a_2 = 2.1 + 1 = 3, a_3 = 2.3 + 1 = 7, a_4 = 2.7 + 1 = 15$, and it appears that $a_n = 2^n - 1$. This can be confirmed by induction, or by iteration:

$$\begin{aligned} a_n &= 1 + 2a_{n-1} = 1 + 2(1 + 2a_{n-2}) = 1 + 2 + 2^2 a_{n-2} \\ &= 1 + 2 + 2^2(1 + 2a_{n-3}) = 1 + 2 + 2^2 + 2^3 a_{n-3} \\ &= 1 + 2 + 2^2 + \cdots + 2^{n-2} + 2^{n-1} a_1 \\ &= 1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1. \end{aligned}$$

In the mythical story attached to the puzzle, n was 64 and priests had to move discs of pure gold; when all was accomplished, the end of the world would come. But $2^{64} - 1 = 18\,446\,744\,073\,709\,551\,615$, and at one move per second the process would take about 5.82×10^{11} years; so we have nothing to worry about! This is another good example of combinatorial explosion. ∎

Example 2.2

There are 3^n n-digit sequences in which each digit is $0, 1$ or 2. How many of these sequences have an **odd** number of 0s?

Solution

Let b_n denote the number of such sequences of length n with an odd number of 0s. Each such sequence ends in $0, 1$ or 2. A sequence ending in 1 has any of the b_{n-1} sequences of length $n-1$ preceding the 1; and similarly there are b_{n-1} sequences ending in 2. If a sequence ends in 0, the 0 must be preceded by a sequence of length $n-1$ with an **even** number of 0s; but the number of such sequences is 3^{n-1} (the total number of sequences of length $n-1$) minus b_{n-1} (the number of sequences of length $n-1$ with an **odd** number of 0s); thus there are $3^{n-1} - b_{n-1}$ sequences ending in 0. So, by the addition principle,

$$b_n = b_{n-1} + b_{n-1} + 3^{n-1} - b_{n-1} \quad \text{i.e. } b_n = b_{n-1} + 3^{n-1}.$$

Again we can find b_n by iteration:

$$\begin{aligned} b_n &= 3^{n-1} + b_{n-1} = 3^{n-1} + (3^{n-2} + b_{n-2}) = \ldots \\ &= 3^{n-1} + 3^{n-2} + \cdots + 3^1 + b_1. \end{aligned}$$

But $b_1 = 1$ (why?), so

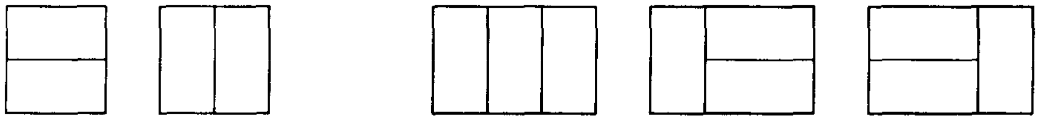
$$b_n = 1 + 3 + \cdots + 3^{n-1} = \frac{1}{2}(3^n - 1).$$

Example 2.3 (Paving a garden path)

A path is 2 metres wide and n metres long. It is to be paved using paving stones of size 1m x 2m. In how many ways can the paving be accomplished?

Solution

Let p_n denote the number of pavings of a $2 \times n$ path. Clearly $p_1 = 1$ since one paving stone fills the path. Also, $p_2 = 2$, the two possibilities being shown in Figure 2.1(a), and $p_3 = 3$ (Figure 2.1(b)).

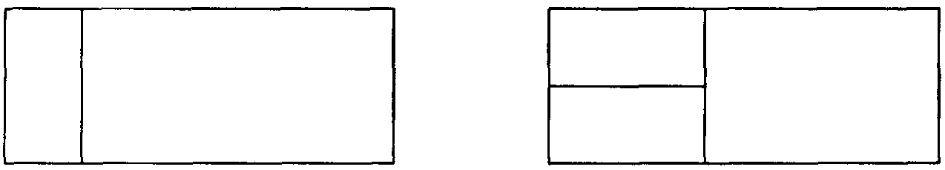


(a) (b)

Figure 2.1

It might appear that $p_n = n$ for all n, but check now that $p_4 = 5$. What is p_n?

For a $2 \times n$ path, the paving must start with one of the options shown in Figure 2.2.



1 $n-1$ **2** $n-2$

Figure 2.2

In the first case it can be completed in p_{n-1} ways; in the second it can be completed in p_{n-2} ways. So, again by the addition principle,

$$p_n = p_{n-1} + p_{n-2} \qquad (n \geq 3).$$

This is a **second order** recurrence relation, since each p_n is given in terms of the previous two. We obtain $p_5 = 5 + 3 = 8$, $p_6 = 8 + 5 = 13, p_7 = 13 + 8 = 21$, etc; the sequence (p_n) thus turns out to be the well-known **Fibonacci sequence** (F_n):

$$1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots.$$

Fibonacci, or Leonardo of Pisa (c. 1200 AD) introduced this sequence when investigating the growth of the rabbit population (see Exercise 2.5); it crops up amazingly frequently in diverse mathematical situations. We shall obtain a formula for F_n in the next section.

Example 2.4 (Flags)

A flag is to consist of n horizontal stripes, where each stripe can be any one of red, white and blue, no two adjacent stripes having the same colour. Under these conditions, the first (top) stripe can be any of three colours, the second has two possibilities, the third has two, and so on (each stripe avoiding the colour of the one above it); so there are $3 \times 2^{n-1}$ possible designs.

Suppose now that, in order to avoid possible confusion of flying the flag upside-down, it is decreed that the top and bottom stripes should be of different colours. Let a_n denote the number of such flags with n stripes. Then $a_1 = 0$ (why?) and $a_2 = 6$. Further, since there is a one-to-one correspondence between flags of n stripes with bottom stripe same as top, and flags of $n-1$ stripes with bottom stripe different from top,

$$\begin{aligned} a_n &= 3 \times 2^{n-1} - (\text{no. of flags with bottom colour same as top colour}) \\ &= 3 \times 2^{n-1} - (\text{no. of flags of } n-1 \text{ stripes with bottom colour} \\ &\qquad\qquad \text{different from top}). \end{aligned}$$

Thus

$$a_n = 3.2^{n-1} - a_{n-1}. \tag{2.1}$$

We could iterate again (try it!), but here is another method. Since

$$a_n + a_{n-1} = 3.2^{n-1}$$

we also have

$$a_{n-1} + a_{n-2} = 3.2^{n-2}$$

whence

$$2(a_{n-1} + a_{n-2}) = 3.2^{n-1} = a_n + a_{n-1}.$$

Thus

$$a_n = a_{n-1} + 2a_{n-2}. \tag{2.2}$$

This again is a second order recurrence relation; we now show how to solve it.

2.2 The Auxiliary Equation Method

In this section we concentrate on recurrence relations of the form

$$a_n = Aa_{n-1} + Ba_{n-2} \qquad (n \geq 3) \tag{2.3}$$

where A, B are constants, $B \neq 0$, and where a_1 and a_2 are given. Equation (2.3) is called a second order linear recurrence relation with constant coefficients; it turns out that there is a very neat method of solving such recurrences.

First, we ask: are there any real numbers $\alpha \neq 0$ such that $a_n = \alpha^n$ satisfies (2.3)? Substituting $a_n = \alpha^n$ into (2.3) gives $\alpha^n = A\alpha^{n-1} + B\alpha^{n-2}$, i.e. $\alpha^2 = A\alpha + B$. Thus $a_n = \alpha^n$ is a solution of (2.3) precisely when α is a solution of the **auxiliary equation**

$$x^2 = Ax + B. \tag{2.4}$$

Thus if α and β are distinct roots of (2.4), $a_n = \alpha^n$ and $a_n = \beta^n$ both satisfy (2.3). If the auxiliary equation has a repeated root α, then

$$x^2 - Ax - B = (x - \alpha)^2 = x^2 - 2\alpha x + \alpha^2$$

so that $A = 2\alpha$ and $B = -\alpha^2$. In this case $a_n = n\alpha^n$ also satisfies (2.3), since

$$\begin{aligned} Aa_{n-1} + Ba_{n-2} &= A(n-1)\alpha^{n-1} + B(n-2)\alpha^{n-2} \\ &= 2(n-1)\alpha^n - (n-2)\alpha^n = n\alpha^n = a_n. \end{aligned}$$

We now prove

Theorem 2.1

Suppose (a_n) satisfies (2.3), and that a_1 and a_2 are given. Let α, β be the roots of the auxiliary equation (2.4). Then

(i) if $\alpha \neq \beta$, there are constants K_1, K_2 such that $a_n = K_1\alpha^n + K_2\beta^n$ for all $n \geq 1$;

(ii) if $\alpha = \beta$, there are constants K_3, K_4 such that $a_n = (K_3 + nK_4)\alpha^n$ for all $n \geq 1$.

Proof

(i) Choose K_1, K_2 so that $a_1 = K_1\alpha + K_2\beta, a_2 = K_1\alpha^2 + K_2\beta^2$, i.e. take

$$K_1 = \frac{a_1\beta - a_2}{\alpha(\beta - \alpha)}, \quad K_2 = \frac{a_1\alpha - a_2}{\beta(\alpha - \beta)}. \tag{2.5}$$

Then the assertion that $a_n = K_1\alpha^n + K_2\beta^n$ is certainly true for $n = 1, 2$. We now proceed by induction. Assume the assertion is true for all $n \leq k$. Then

$$\begin{aligned} a_{k+1} &= Aa_k + Ba_{k-1} = A(K_1\alpha^k + K_2\beta^k) + B(K_1\alpha^{k-1} + K_2\beta^{k-1}) \\ &= K_1\alpha^{k-1}(A\alpha + B) + K_2\beta^{k-1}(A\beta + B) \\ &= K_1\alpha^{k+1} + K_2\beta^{k+1}, \end{aligned}$$

so the result follows.

(ii) Choose K_3, K_4 so that $A_1 = (K_3 + K_4)\alpha$, $a_2 = (K_3 + 2K_4)\alpha^2$, i.e. take

$$K_3 = \frac{2a_1\alpha - a_2}{\alpha^2}, \quad K_4 = \frac{a_2 - a_1\alpha}{\alpha^2}. \tag{2.6}$$

Then the assertion that $a_n = (K_3 + nK_4)\alpha^n$ is certainly true for $n = 1, 2$. Assume it is true for all $n \leq k$. Then

$$\begin{aligned} a_{k+1} &= Aa_k + Ba_{k-1} = A(K_3 + kK_4)\alpha^k + B(K_3 + (k-1)K_4)\alpha^{k-1} \\ &= K_3\alpha^{k-1}(A\alpha + B) + K_4\alpha^{k-1}(Ak\alpha + B(k-1)) \\ &= K_3\alpha^{k+1} + K_4\alpha^{k-1}(2k - \alpha^2(k-1)) \\ &= K_3\alpha^{k+1} + K_4(k+1)\alpha^{k+1}, \end{aligned}$$

as required.

Example 2.4 (continued)

In the flag problem we obtained the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$, where $a_1 = 0, a_2 = 6$. The auxiliary equation $x^2 - x - 2 = 0$ has solutions $\alpha = -1, \beta = 2$, so

$$a_n = K_1(-1)^n + K_2 2^n$$

where $0 = -K_1 + 2K_2$ and $6 = K_1 + 4K_2$, i.e. $K_1 = 2, K_2 = 1$. So

$$a_n = 2(-1)^n + 2^n.$$

Example 2.3 (continued)

The **Fibonacci sequence** (F_n) is given by

$$F_1 = 1,\ F_2 = 2,\ F_n = F_{n-1} + F_{n-2} \quad (n \geq 3).$$

The auxiliary equation $x^2 - x - 1 = 0$ has solutions $\frac{1}{2}(1 \pm \sqrt{5})$, so

$$F_n = K_1\alpha^n + K_2\beta^n$$

where $\alpha = \frac{1}{2}(1+\sqrt{5})$, $\beta = \frac{1}{2}(1-\sqrt{5})$. The initial condition $F_1 = 1, F_2 = 2$, along with (2.5), yield $K_1 = \frac{\alpha}{\sqrt{5}}$, $K_2 = \frac{-\beta}{\sqrt{5}}$, so that

$$F_n = \frac{1}{\sqrt{5}}\alpha^{n+1} - \frac{1}{\sqrt{5}}\beta^{n+1} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}. \quad (2.7)$$

This result may seem rather odd since F_n is to be an integer. Check that expansion by the binomial theorem leads to a cancellation of all terms involving $\sqrt{5}$, giving

$$F_n = \frac{1}{2^n}\left\{\binom{n+1}{1} + 5\binom{n+1}{3} + 5^2\binom{n+1}{5} + \cdots\right\}.$$

This again is a surprise since it is by no means obvious that the sum of binomial coefficients should be divisible by 2^n.

Note that, since $|\beta| < 1$, the second term in (2.7) tends to 0 as $n \to \infty$, giving

$$\frac{F_{n+1}}{F_n} \to \frac{1+\sqrt{5}}{2}, \text{ the golden ratio.}$$

Example 2.5

Solve the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$ $(n \geq 3), a_1 = 1, a_2 = 3$.

Solution

The auxiliary equation is $x^2 - 4x + 4 = 0$, i.e. $(x-2)^2 = 0$, so

$$a_n = (K_1 + nK_2)2^n.$$

The initial conditions give $1 = 2(K_1 + K_2)$, $3 = 4(K_1 + 2K_2)$, whence $K_1 = K_2 = \frac{1}{4}$. Thus

$$a_n = (n+1)2^{n-2}.$$

The auxiliary equation method extends to higher order recurrences in the obvious way.

Example 2.6

Suppose that $a_1 = 3, a_2 = 6, a_3 = 14$ and, for $n \geq 4$,

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}.$$

Then the auxiliary equation is $x^3 - 6x^2 + 11x - 6 = 0$, i.e. $(x-1)(x-2)(x-3) = 0$, so $a_n = K_1 + K_2 2^n + K_3 3^n$. Using the initial conditions, we get

$$a_n = 1 + 2^{n-1} + 3^{n-1}.$$

Non-homogeneous recurrence relations

The auxiliary equation method has been used for recurrence relations such as $a_n = a_{n-1} + 2a_{n-2}$. These are **homogeneous** linear recurrences with constant coefficients: a_n is a linear combination of some of the previous a_i. We now briefly consider the **non-homogeneous** case, e.g.

$$a_n = Aa_{n-1} + Ba_{n-2} + t_n,$$

where t_n is some function of n. One example of this was (2.1), which we solved by manipulating it into a second order homogeneous recurrence; but now we give an alternative method of solution. For we can obtain a solution by first finding the solution of the recurrence relation obtained by replacing t_n by 0, and then adding to it **any** particular solution of the non-homogeneous recurrence.

Example 2.4 (again)

We solve $a_n = -a_{n-1} + 3.2^{n-1}$, $a_1 = 0$.

Solution

First we solve $a_n = -a_{n-1}$. We could use the auxiliary equation $x = -1$, but it is easy just to spot that $a_n = (-1)^{n-1}a_1$, i.e. $a_n = K(-1)^n$. For a particular solution of $a_n = -a_{n-1} + 3.2^{n-1}$, we try something sensible such as $a_n = A2^n$. Substituting gives $A2^n = -A2^{n-1} + 3.2^{n-1}$, whence $A = 1$. So we have $a_n = K(-1)^n + 2^n$. Since $a_1 = 0$, we need $K = 2$; so we have finally $a_n = 2(-1)^n + 2^n$, as before.

Note that the initial conditions are not applied until the final stage of the procedure.

2.3 Generating Functions

The generating function of a sequence $a_1, a_2, a_3, \ldots$ is defined to be

$$f(x) = \sum_{i=1}^{\infty} a_i x^i.$$

For example, the generating function of the Fibonacci sequence is

$$x + 2x^2 + 3x^3 + 5x^4 + \cdots.$$

If a sequence starts with a_0 we take $f(x) = \sum_{i=0}^{\infty} a_i x^i$; for example, the generating function of the sequence $a_n = 2^n$ $(n \geq 0)$ is

$$1 + 2x + 2^2x^2 + \cdots = \frac{1}{1-2x}.$$

Sometimes, given a recurrence relation, it is possible to find the generating function of the sequence and then to find a_n by reading off the coefficient of x^n.

Example 2.4 (yet again!)

Consider the recurrence relation $a_n = 3.2^{n-1} - a_{n-1}$ $(n \geq 2)$, $a_1 = 0$. Let $f(x) = a_1 x + a_2 x^2 + \cdots$. Then

$$\begin{aligned} f(x) &= a_1 x + (3.2 - a_1)x^2 + (3.2^2 - a_2)x^3 + \cdots \\ &= a_1 x + 3(2x^2 + 2^2 x^3 + \cdots) - (a_1 x^2 + a_2 x^3 + \cdots) \\ &= 0 + 6x^2(1 + 2x + 2^2 x^2 + \cdots) - x f(x). \end{aligned}$$

Thus $(1+x)f(x) = \frac{6x^2}{1-2x}$ so that

$$f(x) = 6x^2 \frac{1}{(1+x)(1-2x)} = 2x^2\left(\frac{2}{1-2x} + \frac{1}{1+x}\right)$$

on using the method of partial fractions. Thus

$$f(x) = 4x^2(1 + 2x + 2^2x^2 + \cdots) + 2x^2(1 - x + x^2 - \cdots).$$

Reading off the coefficient of x^n gives

$$a_n = 4.2^{n-2} + 2(-1)^{n-2} = 2^n + 2(-1)^n,$$

as before.

Example 2.5 (again)

$a_n = 4a_{n-1} - 4a_{n-2}$ $(n \geq 3), a_1 = 1, a_2 = 3.$

$$\begin{aligned} f(x) &= a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ &= x + 3x^2 + (4a_2 - 4a_1)x^3 + (4a_3 - 4a_2)x^4 + \cdots \\ &= x + 3x^2 + 4(a_2 x^3 + a_3 x^4 + \cdots) - 4(a_1 x^3 + a_2 x^4 + \cdots) \\ &= x + 3x^2 + 4x(f(x) - a_1 x) - 4x^2 f(x), \end{aligned}$$

so that

$$(1 - 4x + 4x^2)f(x) = x + 3x^2 - 4x^2 = x - x^2.$$

Thus

$$f(x) = \frac{x - x^2}{(1-2x)^2}.$$

Now, since

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots,$$

differentiating gives

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots,$$

so that

$$\frac{1}{(1-2x)^2} = 1 + 2.2x + 3.2^2x^2 + \cdots.$$

Thus

$$f(x) = (x - x^2)(1 + 2.2x + 3.2^2x^2 + 4.2^3x^3 + \cdots)$$

whence

$$\begin{aligned} a_n = {} & \text{coefficient of } x^{n-1} \text{ in } 1 + 2.2x + \cdots \\ & - \text{ coefficient of } x^{n-2} \text{ in } 1 + 2.2x + \cdots. \\ = {} & n.2^{n-1} - (n-1)2^{n-2} = (n+1)2^{n-2} \end{aligned}$$

as before.

2.4 Derangements

Suppose that n people at a party leave their coats in the cloakroom. After the party, they each take a coat at random. How likely is it that no person gets the correct coat?

A **derangement** of $1, \ldots, n$ is a permutation π of $1, \ldots, n$ such that $\pi(i) \neq i$ for each i. For example, there are nine derangements of $1, 2, 3, 4$:

2	4	1	3
2	1	4	3
2	3	4	1
3	1	4	2
3	4	2	1
3	4	1	2
4	1	2	3
4	3	1	2
4	3	2	1

In each of these 1 is not in the first place, 2 is not in the second, and so on. Let d_n denote the number of derangements of $1, \ldots, n$. Then (check!)

$$d_1 = 0, \quad d_2 = 1, \quad d_3 = 2, \quad d_4 = 9.$$

Our aim is to obtain a recurrence relation for the d_i and then use it to obtain a formula for d_n. Before proceeding to the recurrence relation, note that d_n is the number of ways of assigning n objects to n boxes, where, for each object, there is one prohibited box, and where each box is prohibited to just one object. Above, the objects and boxes are both labelled by $1, \dots, n$, with box (position) i prohibited to object (number) i, but the labelling of the boxes and objects is of course arbitrary and does not affect the problem.

Next note that in three of the nine derangements of $1, 2, 3, 4$ listed above, 4 swaps places with another number: this happens in 2143, 3412 and 4321. In the remaining derangements 4 does **not** swap places with another. With this in mind, we put

$$d_n = e_n + f_n$$

where e_n, f_n denote the numbers of derangements of $1, \dots, n$ in which n swaps, does not swap, places with another. Now if n swaps places with i (and there are $n-1$ possible choices for i), the remaining $n-2$ numbers have to be deranged, and this can be done in d_{n-2} ways; so

$$e_n = (n-1)d_{n-2}.$$

If n does not swap places with any other, then some r goes to place n (and there are $n-1$ choices of r), while n does not go to place r. So we have to assign places to $1, \dots, n$, excluding r, where the places available are $1, \dots, n-1$, and where each has precisely one forbidden place (for $i \neq r, n$, place i is forbidden; for $i = n$, place r is forbidden). So there are d_{n-1} possible arrangements, and so

$$f_n = (n-1)d_{n-1}.$$

Thus by the addition principle, we have

$$\boxed{d_n = (n-1)(d_{n-1} + d_{n-2}).} \tag{2.8}$$

Using this recurrence we get

$$d_5 = 4(9+2) = 44, \qquad d_6 = 5(44+9) = 265,$$

and so on.

The recurrence (2.8) does not permit the use of the auxiliary equation method, since the coefficients of d_{n-1} and d_{n-2} are not constants. However, we can manipulate (2.8) into a more manageable form. Equation (2.8) can be rewritten as

$$d_n - nd_{n-1} = -(d_{n-1} - (n-1)d_{n-2}),$$

where the expression on the right is the negative of the expression on the left, with n replaced by $n-1$. So iteration gives

$$\begin{aligned} d_n - nd_{n-1} &= -(d_{n-1} - (n-1)d_{n-2}) \\ &= (-1)^2(d_{n-2} - (n-2)d_{n-3}) \end{aligned}$$

$$\vdots$$
$$= (-1)^{n-2}(d_2 - 2d_1) = (-1)^n(1-0) = (-1)^n,$$

i.e.

$$\boxed{d_n - nd_{n-1} = (-1)^n.} \tag{2.9}$$

Thus

$$\frac{d_n}{n!} - \frac{d_{n-1}}{(n-1)!} = \frac{(-1)^n}{n!}.$$

If we now sum the identities

$$\frac{d_m}{m!} - \frac{d_{m-1}}{(m-1)!} = \frac{(-1)^m}{m!}$$

over $m = 2, 3, \ldots, n$, we get cancellations on the left, giving

$$\frac{d_n}{n!} - \frac{d_1}{1!} = \frac{(-1)^2}{2!} + \frac{(-1)^3}{3!} + \cdots + \frac{(-1)^n}{n!} = \sum_{m=2}^{n} \frac{(-1)^m}{m!} = \sum_{m=0}^{n} \frac{(-1)^m}{m!}.$$

But $d_1 = 0$, so we obtain

$$d_n = n! \sum_{m=0}^{n} \frac{(-1)^m}{m!} = n!\{1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{(-1)^n}{n!}\}. \tag{2.10}$$

One interesting consequence of (2.10) is that, as $n \to \infty$,

$$\frac{d_n}{n!} \to \frac{1}{e},$$

so the probability of no one getting their own coat back after the party tends to $\frac{1}{e} = 0.367\,88$ as $n \to \infty$. Indeed, for n as small as 6,

$$\frac{d_6}{6!} = \frac{265}{720} = 0.368\,06,$$

agreeing with $\frac{1}{e}$ to 3 decimal places.

Example 2.7

(a) Find the number of permutations of $1, \ldots, n$ in which exactly k of the numbers are in their correct position, and deduce that

$$n! = \sum_{\ell=0}^{n} \binom{n}{\ell} d_\ell. \tag{2.11}$$

(b) What is the average number of numbers in their correct position in a random permutation of $1, \cdots, n$?

Solution

(a) There are $\binom{n}{k}$ ways of choosing the k numbers to be fixed. The remaining $n-k$ have to be deranged, and this can be done in d_{n-k} ways. So there are $\binom{n}{k}d_{n-k}$ permutations with exactly k fixed numbers.

But any of the $n!$ permutations fixes k numbers for some k between 0 and n. So

$$n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k} = \sum_{\ell=0}^{n} \binom{n}{\ell} d_\ell$$

on putting $\ell = n-k$.

(b) The average number of fixed numbers in a permutation of $1, \ldots, n$ is

$$\frac{1}{n!}\sum_{k=0}^{n} k\binom{n}{k} d_{n-k} = \frac{1}{n!}\sum_{k=1}^{n} k\binom{n}{k} d_{n-k}$$

$$= \frac{1}{n!}\sum_{k=1}^{n} n\binom{n-1}{k-1} d_{n-k} \text{ by (1.2)}$$

$$= \frac{1}{(n-1)!}\sum_{k=1}^{n} \binom{n-1}{n-k} d_{n-k}$$

$$= \frac{1}{(n-1)!}\sum_{\ell=0}^{n-1} \binom{n-1}{\ell} d_\ell \text{ (on putting } \ell = n-k)$$

$$= \frac{1}{(n-1)!}(n-1)! \qquad \text{by(2.11)}$$

$$= 1.$$

So the average number of fixed numbers is 1.

Alternative proofs of (2.10)

A proof of (2.10) using the inclusion-exclusion principle will be given in Chapter 6. Here we give yet another proof, a simple application of the inversion principle, as in Corollary 1.15, applied to (2.11).

In (2.11), put a_n for $n!$ and b_n for d_n. Then (2.11) is

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k,$$

so that, by Corollary 1.15,

$$d_n = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} k! = \sum_{k=0}^{n} (-1)^{n+k} \frac{n!}{(n-k)!}$$

$$= n!\sum_{\ell=0}^{n}\frac{(-1)^{2n-\ell}}{\ell!} \qquad \text{(on putting } \ell = n-k\text{)}$$

$$= n!\sum_{\ell=0}^{n}\frac{(-1)^{\ell}}{\ell!}.$$

2.5 Sorting Algorithms

Given a pile of exam scripts, we might want to **sort** them, i.e. put them in increasing or decreasing order of marks. Are there any efficient ways of doing this? We start with a simple but not very efficient procedure.

Bubblesort

Take a list of n numbers, in random order. Compare the first two, swapping them round if they are not in increasing order. Then compare the second and third numbers, again swapping if necessary. In this way proceed up the sequence; the largest number will then be at the end. Next repeat the whole process for the first $n-1$ numbers: this will take the second largest to the second last position. Repeat for the first $n-2$, and so on.

The total number of comparisons involved in this procedure is

$$(n-1)+(n-2)+\cdots+2+1 = \frac{1}{2}n(n-1) = \frac{1}{2}n^2 - \frac{1}{2}n,$$

so we say that the bubblesort algorithm has $O(n^2)$ complexity.

Example 2.8

Start with $7, 10, 4, 6, 3$.

After the first 4 comparisons we have $7, 4, 6, 3, 10$.
After the next 3 comparisons we have $4, 6, 3, 7, 10$.
After the next 2, we have $4, 3, 6, 7, 10$.
After the final comparison we have $3, 4, 6, 7, 10$.

Mergesort

The idea here is to split the given list into two (roughly) equal parts, sort each separately, and then merge (combine) them.

The process of combining two sorted lists of lengths ℓ and m into one list can be accomplished by $\ell + m - 1$ comparisons. For suppose we have two such lists, both in increasing order. Compare the first (smallest) numbers in the lists, and take the smaller as the first member of a new list L, crossing it out of its original position. Repeat the process to find the second member of L, and so on. The number of comparisons is clearly $\ell + m - 1$, since when only one number from the two original lists is left no comparison is necessary.

Before the merging takes place, the two halves of the original list can be sorted by a similar method. Let t_n denote the number of comparisons needed to sort a list of n members by this method. If we split n into $\ell + k$, then $t_n = t_\ell + t_k + \ell + k - 1 = t_\ell + t_k + n - 1$.

Thus, if we consider the particular case where $n = 2^m$, so that the lists can be bisected at each stage, we have

$$t_{2^m} = 2t_{2^{m-1}} + (2^m - 1).$$

Put $a_m = t_{2^m}$; then the recurrence relation becomes

$$a_m = 2a_{m-1} + (2^m - 1). \qquad (2.12)$$

Using the method of Section 2.2, first solve the homogeneous recurrence $a_m = 2a_{m-1}$. The solution is clearly $a_n = A2^n$ for some constant A. We then have to find a particular solution of (2.12). Try

$$a_n = Bn2^n + C.$$

(Trying $a_n = B.2^n + C$ would not work, since $a_n = 2^n$ is already a solution of the homogeneous recurrence; so we take the hint given by Theorem 2.1(ii) and insert n.) We then require

$$Bn2^n + C = 2B(n-1)2^{n-1} + 2C + 2^n - 1$$

i.e.

$$0 = -B.2^n + 2^n - 1 + C.$$

So take $B = C = 1$ to obtain finally $a_n = A.2^n + n2^n + 1$. But $a_1 = 1$, so $A = -1$, giving

$$a_n = 2^n(n-1) + 1.$$

Thus $t_{2^m} = 1 + 2^m(m-1)$. On putting $n = 2^m$, we get

$$t_n = 1 + n(\log_2 n - 1),$$

so the mergesort method has complexity $O(n \log n)$, an improvement on the $O(n^2)$ of bubblesort.

2.6 Catalan Numbers

In this section we introduce a well-known sequence of numbers known as the Catalan numbers, which arise as the counting numbers of a remarkable number of different types of structure. They are named after the Belgian mathematician E.C. Catalan (1814–1894) who discussed them in his publications, but they had been studied earlier by several mathematicians, including Euler in his work on triangulating polygons (to be discussed shortly).

We describe fully one of the occurrences of Catalan numbers, and begin with the following easy problem.

Example 2.9

How many "up-right" routes are there from A to B in Figure 2.3?

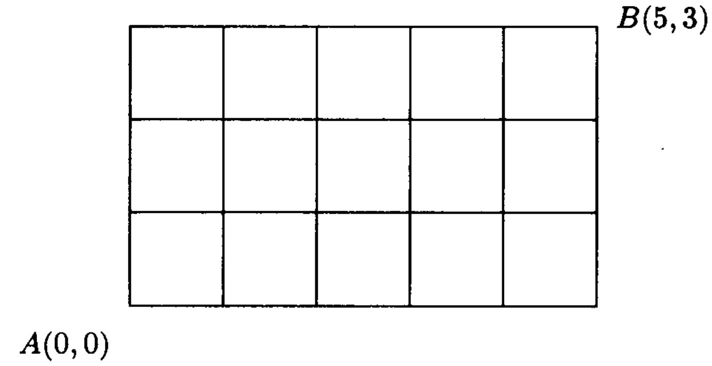


Figure 2.3

Solution

By an "up-right" route we mean a path from A to B following edges of squares, always moving up or to the right. Any path must consist of 8 moves, 5 of which must be to the right, and 3 up. So the total number of possible routes is $\binom{8}{3}$.

More generally, the number of up-right routes from the bottom left vertex to the top right vertex of an $m \times n$ array is $\binom{m+n}{n}$.

Suppose we now have a **square** $n \times n$ array, and ask for the number p_n of up-right paths from bottom left to top right **which never go above the diagonal** AB. In the case $n = 3$, shown in Figure 2.4, there are 5 such routes represented by *RURURU, RURRUU, RRUURU, RRURUU, RRRUUU* where R, U stand respectively for right, up. Thus $p_3 = 5$. What is p_n?

Any qualifying route (let's call it a **good** route) from A to B must "hit" the diagonal at some stage before B, even if it is only at A. So consider any good route from A to B, and suppose that, prior to reaching B, it **last** met the diagonal at the point $C(m, m)$ where $1 \leq m < n$. Then there are p_m possibilities for the part of the route between A and C. The route must then proceed to $D(m+1, m)$, and eventually to $E(n, n-1)$, but it must **never** go

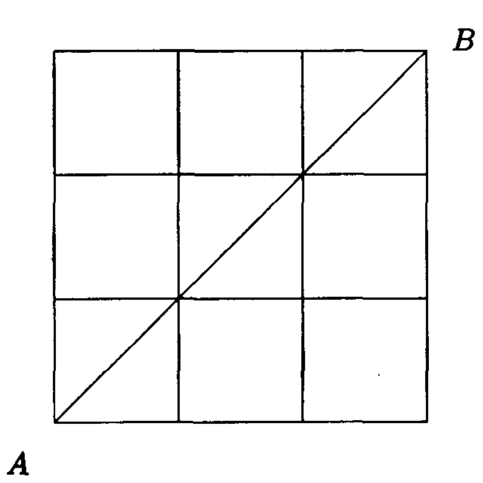


Figure 2.4

above the line DE, since otherwise C would not have been the last hit before B. But D and E are opposite vertices of a square of side length $n-m-1$, so there are p_{n-m-1} good routes from D to E. See Figure 2.5.

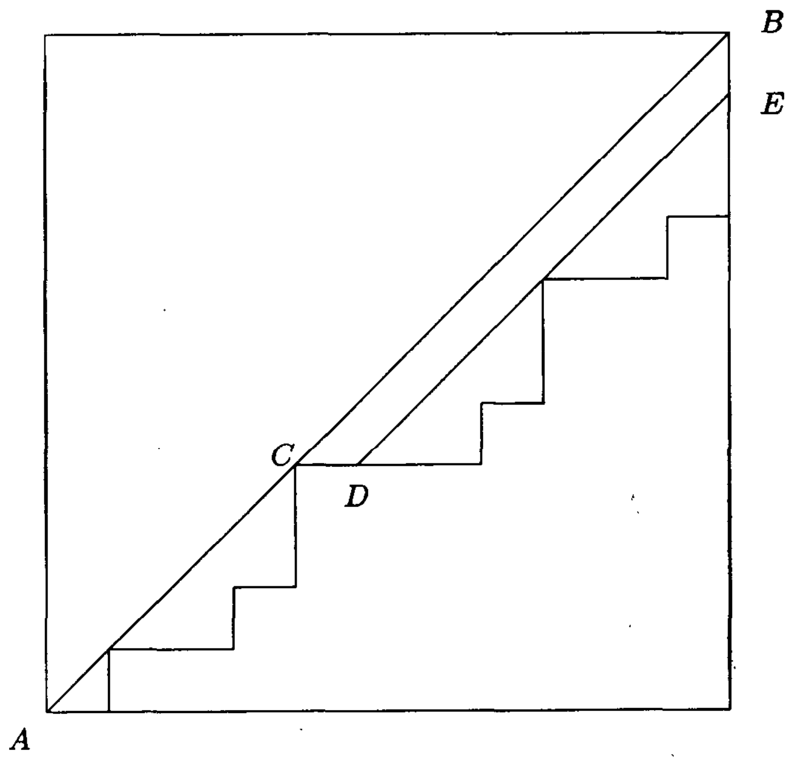


Figure 2.5

By the multiplication principle, the number of good routes from A to B, with (m,m) as the last contact with the diagonal before B, is therefore $p_m p_{n-m-1}$.

Since m can take any value from 0 to $n-1$, it now follows from the addition principle that, with $p_0 = 1$,

$$p_n = \sum_{m=0}^{n-1} p_m p_{n-m-1}. \tag{2.13}$$

This recurrence relation differs from the ones met so far, but we can use generating functions to solve it. Let $f(x)$ be the generating function:

$$f(x) = p_0 + p_1 x + p_2 x^2 + \cdots.$$

Then

$$\begin{aligned} f^2(x) &= (p_0 + p_1 x + p_2 x^2 + \cdots)(p_0 + p_1 x + p_2 x^2 + \cdots) \\ &= \sum_{n=0}^{\infty} x^n (p_0 p_n + p_1 p_{n-1} + \cdots + p_n p_0) \\ &= \sum_{n=0}^{\infty} p_{n+1} x^n \qquad \text{by (2.13).} \end{aligned}$$

Thus $xf^2(x) = \sum_{n=0}^{\infty} p_{n+1} x^{n+1} = f(x) - 1$, whence

$$xf^2(x) - f(x) + 1 = 0.$$

Solving this quadratic, we obtain

$$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x} = \frac{1}{2x}\{1 - (1-4x)^{\frac{1}{2}}\}.$$

We have to take the negative sign to avoid having a term of the form $\frac{1}{x}$ in $f(x)$. So

$$\begin{aligned} f(x) &= \frac{1}{2x}\{1 - (1 - \frac{1}{2} \cdot 4x - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{4^2 x^2}{2!} - \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{4^3 x^3}{3!} - \cdots)\} \\ &= \frac{1}{2x}\{\frac{1}{2} \cdot 4x + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{4^2 x^2}{2!} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{4^3 x^3}{3!} + \cdots\} \\ &= 1 + \frac{1}{2} \cdot \frac{4x}{2!} + \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{4^2 x^2}{3!} + \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{4^3 x^3}{4!} + \cdots. \end{aligned}$$

Thus, for $n \geq 1$,

$$\begin{aligned} p_n &= \frac{1.3.5.\ldots(2n-1)}{2^n (n+1)!} 4^n = \frac{2^n}{(n+1)!}.1.3.5.\ldots(2n-1) \\ &= \frac{2^n}{(n+1)!} \cdot \frac{(2n)!}{2^n . n!} = \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

Thus, for example, $p_3 = \frac{1}{4}\binom{6}{3} = 5$ and $p_4 = \frac{1}{5}\binom{8}{4} = 14$. Note also that $p_0 = 1$ fits in with the convention that $\binom{0}{0} = 1$.

The numbers p_n are the **Catalan numbers**, usually denoted by C_n. Thus

$$\boxed{C_n = \frac{1}{n+1}\binom{2n}{n}.} \tag{2.14}$$

The sequence $(C_n)_{n\geq 0}$ begins

$$1, 1, 2, 5, 14, 42, 139, 429, \ldots .$$

From (2.13) we have

$$C_m = C_0C_{m-1} + C_1C_{m-2} + \cdots + C_{m-1}C_0. \tag{2.15}$$

As remarked earlier, the Catalan numbers appear in many situations. One immediate interpretation, obtained by replacing R and U by 0 and 1 respectively, is:

C_n = number of binary sequences of length $2n$ containing exactly n 0s and n 1s, such that at each stage in the sequence the number of 1s up to that point never exceeds the number of 0s.

Euler's interest was in the following:

C_{n-2} = number of ways of dividing a convex n-gon into triangles by drawing $n-3$ non-intersecting diagonals. For example, the C_3 ways of triangulating a pentagon are shown in Figure 2.6.

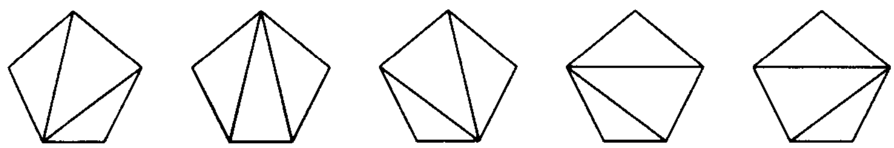


Figure 2.6

See Exercise 2.16 for this problem and Exercise 2.17 for another appearance of C_n.

Another derivation of the formula (2.14)

We close this section by pointing out that there is an alternative ingenious method of counting good up-right routes, due to D. André (1887). It avoids the rather awkward recurrence relation (2.13), instead making use of a clever mirror principle.

The number of good routes from $A(0,0)$ to $B(n,n)$ which do not cross the diagonal AB is the total number $\binom{2n}{n}$ of up-right routes from A to B minus the number of routes which do cross AB. Let's call routes which cross AB **bad** routes.

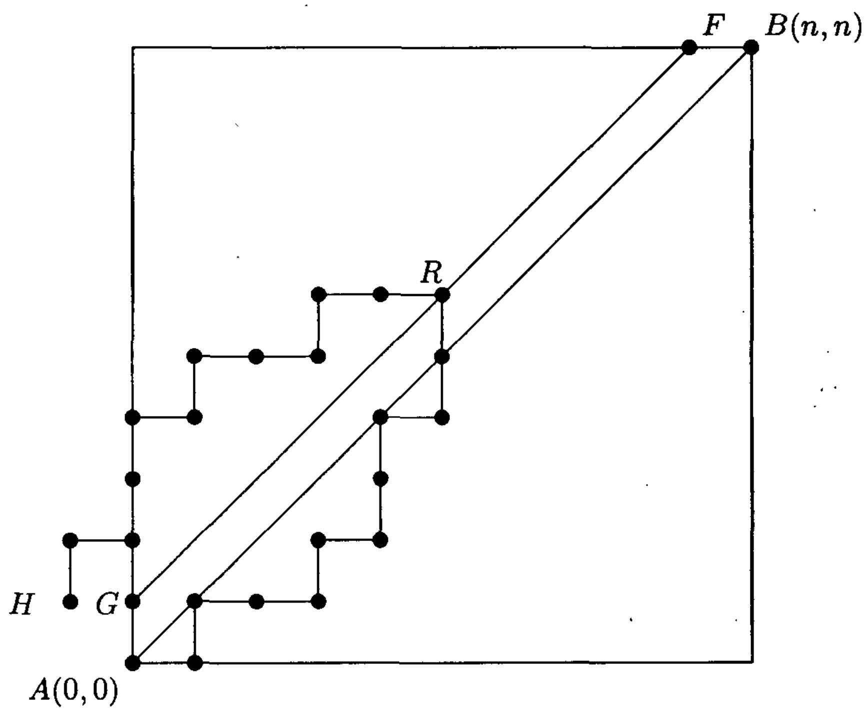


Figure 2.7

Consider any bad route. There will be a **first** point on that route above the diagonal AB; suppose this is the point $R(m, m+1)$. If we replace the part of the route from A to R by its image in the "mirror" GF (see Figure 2.7) then we get an up-right route from $H(-1, 1)$ to $B(n, n)$. Conversely, any up-right route from H to B must cross GF somewhere, and arises from precisely one bad route from A to B. So the number of bad routes is just the number of up-right routes from $(-1, 1)$ to (n, n), which is

$$\binom{n+1\ +\ n-1}{n+1} = \binom{2n}{n+1}.$$

So finally the number of good routes from A to B is

$$\binom{2n}{n} - \binom{2n}{n+1} = \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} = \frac{1}{n+1}\binom{2n}{n}.$$

Exercises

Exercise 2.1

Solve the recurrence relations

(a) $a_n = \frac{1}{2}a_{n-1} + 1, a_1 = 1;$

(b) $a_n = 5a_{n-1} - 6a_{n-2}, a_1 = -1, a_2 = 1$;
(c) $a_n = 6a_{n-1} - 9a_{n-2}, a_1 = 1, a_2 = 9$;
(d) $a_n = 4a_{n-1} - 3a_{n-2} + 2^n, a_1 = 1, a_2 = 11$.

Exercise 2.2

Let b_n denote the number of n-digit binary sequences containing no two consecutive 0s. Show that $b_n = b_{n-1} + b_{n-2}$ $(n \geq 3)$ and hence find b_n.

Exercise 2.3

Let d_n denote the number of n-digit sequences in which each digit is 0, 1 or 2, and containing no two consecutive 1s and no two consecutive 2s. Show that $d_n = 2d_{n-1} + d_{n-2}$. Solve this recurrence and deduce that $d_n = 1 + 2\binom{n+1}{2} + 2^2\binom{n+1}{4} + 2^3\binom{n+1}{6} + \cdots$.

Exercise 2.4

Use generating functions to solve Exercise 2.1(a) and 2.1(b).

Exercise 2.5

Fibonacci's rabbits. Start with 1 pair of rabbits, and suppose that each pair produces one new pair in each of the next two generations and then dies. Find f_n, the number of pairs belonging to the nth generation $(f_1 = 1 = f_2)$.

Exercise 2.6

Solve the recurrence relation (2.1) for the flags by iteration.

Exercise 2.7

The Lucas numbers L_n are defined by $L_1 = 1, L_2 = 3, L_n = L_{n-1} + L_{n-2}$ $(n \geq 3)$. Obtain a formula for L_n.

Exercise 2.8

Solve the recurrence (2.12) by using the method given in Example 2.4, first eliminating 1 and then eliminating powers of 2. You should obtain $a_n - 5a_{n-1} + 8a_{n-2} - 4a_{n-3} = 0$.

Exercise 2.9

Verify that if a'_n and a''_n are two solutions of the recurrence $a_n = Aa_{n-1} + Ba_{n-2}$ then $a'_n + a''_n$ is also a solution.

Exercise 2.10

Show that the generating function for the Fibonacci sequence is $\frac{x(1+x)}{1-x-x^2}$. Hence obtain (2.7).

Exercise 2.11

Let $M = \binom{01}{11}$.

(a) Prove that $M^{n+2} = \begin{pmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{pmatrix}$ where F_n is the nth Fibonacci number.

(b) By taking determinants show that $F_n F_{n+2} - F_{n+1}^2 = (-1)^n$.

(c) By considering the identity $M^{m+n+2} = M^{m+1}M^{n+1}$, prove that $F_{m+n} = F_m F_n + F_{m-1}F_{n-1}$.

Exercise 2.12

Prove that $F_1 + F_2 + \cdots + F_n = F_{n+2} - 2$.

Exercise 2.13

For each of the following, work out the values for the first few values of n and make a guess at the general case. Then prove your guesses by induction.

(a) $F_1 + F_3 + F_5 + \cdots + F_{2n-1}$;

(b) $F_2 + F_4 + F_6 + \cdots + F_{2n}$;

(c) $F_1 - F_2 + F_3 - \cdots + (-1)^{n-1}F_n$.

Exercise 2.14

In bellringing, successive permutations of n bells are played one after the other. Following one permutation π, the next permutation must be obtained from π by moving the position of each bell by at most one place. For example, for $n = 4$, the permutation 1234 could be followed by any one of 2134, 2143, 1324, 1243. Show that if a_n denotes the number of permutations which could follow $12\ldots n$, then $a_n = a_{n-1} + a_{n-2} + 1$. Hence find a_n.

Exercise 2.15

(a) Let g_n denote the number of subsets of $\{1, \ldots, n\}$ containing no two consecutive integers. Thus, for example, $g_1 = 2$ (include the empty set!) and $g_2 = 3$. Find a recurrence relation for g_n, and deduce that $g_n = F_{n+1}$.

(b) A k-element subset of $\{1, \ldots, n\}$ can be considered as a binary sequence of length n containing k 1s and $n - k$ 0s (see Example 1.12). Use Example 1.17 to show that the number of k-subsets of $\{1, \ldots, n\}$ containing no two consecutive integers is $\binom{n-k+1}{k}$.

(c) Deduce that $F_n = \sum_{k \leq \frac{1}{2}n} \binom{n-k}{k}$. How does this relation show up in Pascal's triangle?

Exercise 2.16

Let t_n denote the number of ways of triangulating a convex $(n+2)$-gon by drawing $n - 1$ diagonals. Show that $t_n = C_n$ as follows. Label the vertices $1, \ldots, n+2$, and consider the triangle containing edge 12. If it contains vertex r as its third vertex, in how many ways can the remaining two parts of the interior of the $(n+2)$-gon be triangulated? Deduce that $t_n = \sum t_i t_j$ where summation is over all pairs i, j with $i + j = n - 1$.

Exercise 2.17

Show that if $2n$ points are marked on the circumference of a circle and if a_n is the number of ways of joining them in pairs by n non-intersecting chords, then $a_n = C_n$.

Exercise 2.18

Derive Euler's formula $C_n = \frac{2.4.6\ldots(4n-2)}{(n+1)!}$ for the Catalan numbers, and note that $(n+1)C_n = (4n-2)C_{n-1}$.

Exercise 2.19

Prove that $d_n > (n-1)!$ for all $n \geq 4$.

Exercise 2.20

Insertionsort. Sort a list $x_1, \ldots, x_n$ into increasing order as follows. At stage 1, form list L_1 consisting of just x_1. At stage 2, compare x_1 with x_2 and form list L_2 consisting of x_1 and x_2 in increasing order. At stage

i, when $x_1, \ldots, x_{i-1}$ have been put into list L_{i-1} in increasing order, compare x_i with each x_j in L_{i-1} in turn until its correct position is obtained; this creates list L_i. Repeat until L_n is obtained. Compare the efficiency of this method with that of bubblesort.

Exercise 2.21

In a mathematical model of the population of foxes and rabbits, the populations x_n and y_n of foxes and rabbits at the end of n years are related by $\binom{x_{n+1}}{y_{n+1}} = \begin{pmatrix} 0.6 & 0.5 \\ -0.16 & 1.2 \end{pmatrix} \binom{x_n}{y_n}$.

Show that $5x_{n+2} - 9x_{n+1} + 4x_n = 0$, and hence find x_n in terms of x_0 and y_0.

Deduce that $x_n \to \frac{5}{2}y_0 - x_0$ as $n \to \infty$, provided $x_0 < \frac{5}{2}y_0$. What happens to y_n?

Exercise 2.22

In a football competition, there are n qualifying leagues. At the next stage of the competition, each winner of a league plays a runner up in another league. In how many ways can the winners and the runners up be paired?