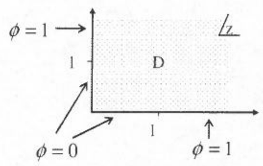
Apellido y Nombres: ....
DNI: .... Padrón: .... Código Asignatura: ....
Cursada. Cuatrimestre: .... Año: .... Profesor: ....
Correo electrónico: ....

Análisis Matemático III.
Examen Integrador. Segunda fecha. 20 de diciembre de 2022.

Justificar claramente todas las respuestas. La aprobación del examen requiere la correcta resolución de 3 (tres) ejercicios

Ejercicio 1. Considerar una placa plana y homogénea que coincide con el primer cuadrante. Formular el problema de la temperatura en estado estacionario en dicha placa con condiciones en la frontera como se indican en la siguiente figura:



y obtener el valor de la temperatura en el punto de coordenadas $(1, 1)$.

Ejercicio 2. Resolver:

$$\begin{cases} u_{xx}+u_{yy}=0 & 0<x<\pi,\ 0<y<1 \\ u(0,y)=u(\pi,y)=1 & 0\leqslant y\leqslant 1 \\ u(x,0)=1+2\operatorname{sen} x+\operatorname{sen}(3x) & 0\leqslant x\leqslant \pi \\ u(x,1)=1+3\ \operatorname{sen}(2x) & 0\leqslant x\leqslant \pi \end{cases}$$

¿Es única la solución?

Ejercicio 3. Sea $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\dfrac{x}{x^4+1}$. Argumentar la existencia de la transformada de Fourier de f y obtenerla.

Ejercicio 4. Resolver, especificando las condiciones supuestas sobre la función f, el problema de la onda en la cuerda semi-infinita:

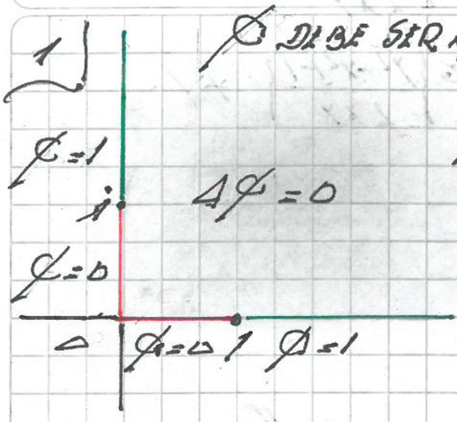
$$\begin{cases} u_{tt}(x,t)=u_{xx}(x,t) & x>0,\quad t>0 \\ u(x,0)=0 & x>0 \\ u_t(x,0)=f(x) & x>0 \\ u(0,t)=0 & t>0 \end{cases}$$

Ejercicio 5. Resolver, aplicando transformada de Laplace:

$$y'(t)+\int_0^t y(u)\,H(t-u)\,du=H(t-1)-H(t-2)$$

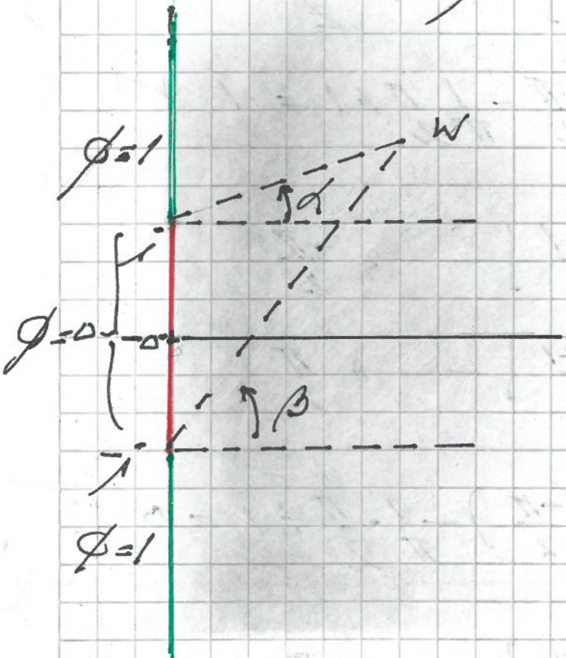
con $y(0^+)=0$ y $H(t)$ función de Heaviside.

1) ϕ DEBE SER ACOTADA (CONSIDERACIONES FÍSICAS) $\therefore$ ES ÚNICA.



$z \longmapsto z^2$

$W = -i z^2$



$\alpha = ARG(W-i)$

$\beta = ARG(W+i)$

$\phi = A\alpha + B\beta + C$

(I) $A\frac{\pi}{2} + B\frac{\pi}{2} + C = 1$

(II) $-A\frac{\pi}{2} + B\frac{\pi}{2} + C = 0 \longleftrightarrow A = \frac{1}{\pi},\ B = \frac{-1}{\pi},\ C = 1$

(III) $-A\frac{\pi}{2} - B\frac{\pi}{2} + C = 1$

$$\phi = \frac{1}{\pi} ARG(W-i) - \frac{1}{\pi} ARG(W+i) + 1 \quad (*)$$

$$W = -i z^2 = -i(x^2-y^2+2xyi) = 2xy - i(x^2-y^2) = 2xy + i(y^2-x^2)$$

$$W - i = 2xy + i(y^2-x^2-1), \quad W+i = 2xy + i(y^2-x^2+1).$$

EN $(*)$:

$$\boxed{\phi(x,y) = \frac{1}{\pi} ARCTG\left[\frac{y^2-x^2-1}{2xy}\right] - \frac{1}{\pi} ARCTG\left[\frac{y^2-x^2+1}{2xy}\right] + 1}$$

OBS: ϕ ES ACOTADA, PUES $|ARCTG(\lambda)| \leq \frac{\pi}{2}$ PARA TODO $\lambda \in \mathbb{R}$.

COMPROBACIÓN DE LAS CONDICIONES DE CONTORNO

$$\phi(x,y) = \frac{1}{\pi} \operatorname{ARTG}\left[\frac{y^2-x^2-1}{2xy}\right] + $$
$$- \frac{1}{\pi} \operatorname{ARTG}\left[\frac{y^2-x^2+1}{2xy}\right] + 1$$

I) $y>1$, $x \to 0^+$ $\left\{\begin{array}{l} y^2-x^2-1>0 \\ y^2-x^2+1>0 \end{array}\right.$:

$$\operatorname{ARTG}\left[\frac{y^2-x^2-1}{2xy}\right] \longrightarrow \frac{\pi}{2}$$
$$\operatorname{ARTG}\left[\frac{y^2-x^2+1}{2xy}\right] \longrightarrow \frac{\pi}{2}$$

$$\therefore \phi \longrightarrow \frac{1}{\pi}\cdot\frac{\pi}{2} - \frac{1}{\pi}\frac{\pi}{2} + 1 = 1 \quad ✓$$

II) $0<y<1$, $x \to 0^+$ $\left\{\begin{array}{l} y^2-x^2-1<0 \\ y^2-x^2+1>0 \end{array}\right.$:

$$\operatorname{ARTG}\left[\frac{y^2-x^2-1}{2xy}\right] \longrightarrow \frac{-\pi}{2}$$
$$\operatorname{ARTG}\left[\frac{y^2-x^2+1}{2xy}\right] \longrightarrow \frac{\pi}{2}$$

$$\therefore \phi \longrightarrow \frac{1}{\pi}\cdot\left(\frac{-\pi}{2}\right) - \frac{1}{\pi}\cdot\frac{\pi}{2} + 1 = 0 \quad ✓$$

III) $0<x<1$, $y \to 0^+$ $\left\{\begin{array}{l} y^2-x^2-1<0 \\ y^2-x^2+1>0 \end{array}\right.$:

$$\operatorname{ARTG}\left[\frac{y^2-x^2-1}{2xy}\right] \longrightarrow -\frac{\pi}{2}$$
$$\operatorname{ARTG}\left[\frac{y^2-x^2+1}{2xy}\right] \longrightarrow \frac{\pi}{2}$$

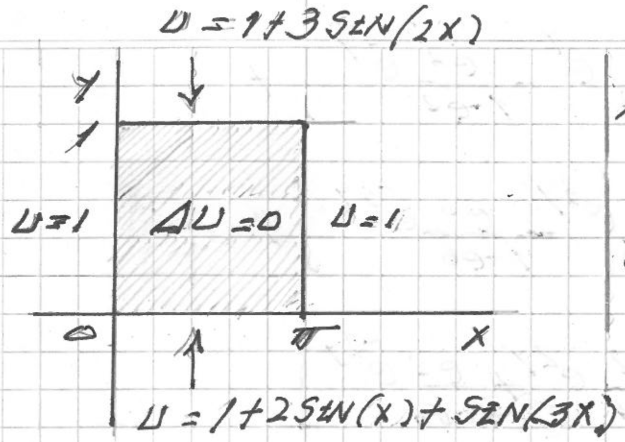
$$\therefore \phi \longrightarrow \frac{1}{\pi}\cdot\left(\frac{-\pi}{2}\right) - \frac{1}{\pi}\cdot\frac{\pi}{2} + 1 = 0 \quad ✓$$

IV) $x>1$, $y \to 0^+$ $\left\{\begin{array}{l} y^2-x^2-1<0 \\ y^2-x^2+1<0 \end{array}\right.$:

$$\operatorname{ARTG}\left[\frac{y^2-x^2-1}{2xy}\right] \longrightarrow \frac{-\pi}{2}$$
$$\operatorname{ARTG}\left[\frac{y^2-x^2+1}{2xy}\right] \longrightarrow \frac{-\pi}{2}$$

$$\therefore \phi \longrightarrow \frac{1}{\pi}\left(\frac{-\pi}{2}\right) - \frac{1}{\pi}\left(\frac{-\pi}{2}\right) + 1 = 1 \quad ✓$$

2)



ALGUNAS ARMÓNICAS BÁSICAS:

$\cos(\omega x)e^{\omega y}$, $\cos(\omega x)e^{-\omega y}$,

$\operatorname{sen}(\omega x)e^{\omega y}$, $\operatorname{sen}(\omega x)e^{-\omega y}$.

$\omega \in \mathbb{R}$

$$U(x,y) = 1 + 2\operatorname{sen}(x)\left[Ae^{y}+Be^{-y}\right] + \operatorname{sen}(3x)\left[Ce^{3y}+De^{-3y}\right] +$$
$$+ 3\operatorname{sen}(2x)\left\{Ee^{2y}+Fe^{-2y}\right\}$$

ES ARMÓNICA Y VERIFICA:

$$U(0,y) = 1 \qquad U(\pi,y) = 1 \quad \checkmark$$

AHORA:

$$U(x,0) = 1+2\operatorname{sen}(x)[A+B] + \operatorname{sen}(3x)[C+D] + 3\operatorname{sen}(2x)[E+F] =$$
$$= 1+2\operatorname{sen}(x)+\operatorname{sen}(3x) \longleftrightarrow$$

(1) $A+B=1$

(2) $C+D=1$

(3) $E+F=0$

$$U(x,1) = 1+2\operatorname{sen}(x)\left[Ae+Be^{-1}\right] + \operatorname{sen}(3x)\left[Ce^{3}+De^{-3}\right] +$$
$$+ 3\operatorname{sen}(2x)\left[Ee^{2}+Fe^{-2}\right]$$
$$= 1+3\operatorname{sen}(2x) \longleftrightarrow$$

(4) $Ae+Be^{-1}=0$

(5) $Ce^{3}+De^{-3}=0$

(6) $Ee^{2}+Fe^{-2}=1$

DE (1) Y (4): $A = \dfrac{1}{1-e^{2}}$, $B = \dfrac{-e^{2}}{1-e^{2}}$

DE (2) Y (5): $C = \dfrac{1}{1-e^{6}}$, $D = \dfrac{-e^{6}}{1-e^{6}}$

DE (3) Y (6): $E = \dfrac{-e^{2}}{1-e^{4}}$, $F = \dfrac{e^{2}}{1-e^{4}}$

$$\therefore \boxed{\begin{aligned} U(x,y) = 1 + 2\,\mathrm{SEN}(x)&\left[\frac{1}{1-e^2}e^{y} - \frac{e^2}{1-e^2}e^{-y}\right] + \\ + \mathrm{SEN}(3x)&\left[\frac{1}{1-e^6}e^{3y} - \frac{e^6}{1-e^6}e^{-3y}\right] + \\ + 3\,\mathrm{SEN}(2x)&\left[\frac{-e^2}{1-e^4}e^{2y} + \frac{e^2}{1-e^4}e^{-2y}\right]\end{aligned}}$$

COMPROBACIÓN: QUE U ES ARMÓNICA Y QUE $U(0,y) = U(\pi,y) = 1$ ES INMEDIATO. AHORA:

$$U(x,0) = 1 + 2\,\mathrm{SEN}(x)\left[\overbrace{\frac{1}{1-e^2} - \frac{e^2}{1-e^2}}^{=1}\right] + $$

$$+ \mathrm{SEN}(3x)\left[\overbrace{\frac{1}{1-e^6} - \frac{e^6}{1-e^6}}^{=1}\right] + 3\,\mathrm{SEN}(2x)\left[\overbrace{\frac{-e^2}{1-e^4} + \frac{e^2}{1-e^4}}^{=0}\right] \checkmark$$

$$U(x,1) = 1 + 2\,\mathrm{SEN}(x)\left[\overbrace{\frac{1}{1-e^2}\cdot e - \frac{e^2}{1-e^2}e^{-1}}^{=e}\right] + $$

$$+ \mathrm{SEN}(3x)\left[\frac{1}{1-e^6}e^{3} - \frac{e^6}{1-e^6}\cdot e^{-3}\right] + $$

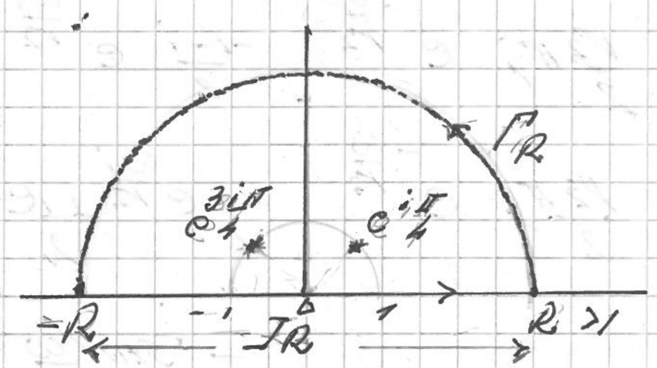
$$+ 3\,\mathrm{SEN}(2x)\underbrace{\left[\frac{-e^2}{1-e^4}\cdot e^2 + \frac{e^2}{1-e^4}e^{-2}\right]}_{\frac{-e^4}{1-e^4} + \frac{1}{1-e^4} = 1} \checkmark$$

3) $f(x) = \frac{x}{x^4+1}$ ES ABSOLUTAMENTE INTEGRABLE, PUES

$|f(x)| = \frac{|x|}{x^4+1}$ ES ASINTÓTICAMENTE EQUIVALENTE A $\frac{1}{x^3}$.

(f ES CONTINUA EN $\mathbb{R}$).

• $$\hat{f}(\omega) = \int_{-\infty}^{+\infty} \frac{x e^{i\omega x}}{x^4+1}\, dx \; :$$



CIRCUITOS PARA $\omega > 0$

$$F(z) = \frac{z e^{i\omega z}}{z^4+1}$$

• $$\int_{I_R} F(z)\,dz + \int_{\Gamma_R} F(z)\,dz = 2\pi i \left\{ Res\left(F, e^{i\pi/4}\right) + Res\left(F, e^{i3\pi/4}\right) \right\}$$

• $$(z - e^{i\pi/4})F(z) = \frac{z - e^{i\pi/4}}{z^4+1} \cdot e^{i\omega z} \xrightarrow{L'H} \frac{1}{4\left(e^{\frac{i\pi}{4}}\right)^3} e^{i\omega e^{\frac{i\pi}{4}}} =$$

$$= \frac{1}{4e^{i\frac{3\pi}{4}}} e^{i\omega\left[1/\sqrt{2} + i/\sqrt{2}\right]} =$$

$$= \frac{1}{4\left[\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right]} e^{i\frac{\omega}{\sqrt{2}}} e^{-\frac{\omega}{\sqrt{2}}} = \frac{\sqrt{2}}{4(-1+i)} e^{-\frac{\omega}{\sqrt{2}}} e^{i\frac{\omega}{\sqrt{2}}} \quad (1)$$

• $$(z - e^{\frac{3i\pi}{4}})F(z) = \frac{z - e^{3i\pi/4}}{z^4+1} \cdot e^{i\omega z} \xrightarrow{L'H} \frac{1}{4\left(e^{\frac{3i\pi}{4}}\right)^3} e^{i\omega e^{\frac{3i\pi}{4}}}$$

$$= \frac{1}{4e^{\frac{9i\pi}{4}}} e^{i\omega\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)} \overset{(*)}{=} \frac{1}{4\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)} e^{-\frac{\omega}{\sqrt{2}}} e^{-\frac{\omega}{\sqrt{2}} i} =$$

$$= \frac{\sqrt{2}}{4(1+i)} \cdot e^{-\frac{\omega}{\sqrt{2}}} e^{-\frac{\omega}{\sqrt{2}} i} \quad (2)$$

(*) $\frac{9\pi}{4} = 2\pi + \frac{\pi}{4}$

$\therefore$

$$2\pi i\left[\operatorname{RES}(F, e^{i\pi/4}) + \operatorname{RES}(F, e^{3i\pi/4})\right] =$$

$$= 2\pi i\left\{\frac{\sqrt{2}}{4(-1+i)}\, e^{-\frac{\omega}{\sqrt{2}}} e^{\frac{\omega i}{\sqrt{2}}} + \frac{\sqrt{2}}{4(1+i)}\, e^{-\frac{\omega}{\sqrt{2}}} e^{-\frac{\omega}{\sqrt{2}} i}\right\} =$$

$$= 2\pi i\,\frac{\sqrt{2}}{4}\, e^{-\frac{\omega}{\sqrt{2}}}\left[\frac{1}{-1+i}\, e^{\frac{\omega i}{\sqrt{2}}} + \frac{1}{1+i}\, e^{-\frac{\omega i}{\sqrt{2}}}\right] =$$

$$= \frac{\sqrt{2}\pi i}{2}\, e^{-\frac{\omega}{\sqrt{2}}}\left[\frac{-1-i}{2}\, e^{\frac{\omega i}{\sqrt{2}}} + \frac{1-i}{2}\, e^{-\frac{\omega i}{\sqrt{2}}}\right] =$$

$$= \frac{\sqrt{2}\pi i}{2}\, e^{-\frac{\omega}{\sqrt{2}}}\left[\underbrace{-\frac{1}{2} e^{\frac{\omega i}{\sqrt{2}}} + \frac{1}{2} e^{-\frac{\omega i}{\sqrt{2}}}}_{-i \operatorname{SEN}\left(\frac{\omega}{\sqrt{2}}\right)} - \underbrace{\frac{i}{2} e^{\frac{\omega i}{\sqrt{2}}} - \frac{i}{2} e^{-\frac{\omega i}{\sqrt{2}}}}_{-i\cos\left(\frac{\omega}{\sqrt{2}}\right)}\right] =$$

$$= \frac{\sqrt{2}\pi}{2}\, e^{-\frac{\omega}{\sqrt{2}}}\left[\cos\left(\frac{\omega}{\sqrt{2}}\right) + \operatorname{SEN}\left(\frac{\omega}{\sqrt{2}}\right)\right] = \int_{-\infty}^{\infty} \frac{x\, e^{i\omega x}}{x^4+1}\, dx$$

PARA $\omega > 0$ (VER PAGINA SIGUIENTE)

$\forall \omega > 0$: (VER CIRCUITOS PAG. 5)

- $\displaystyle\int_{\Sigma R} f(z)\,dz = \int_{-R}^{R} \frac{x e^{i\omega x}}{x^4+1}\,dx \xrightarrow[R\to+\infty]{} \int_{-\infty}^{+\infty} \frac{x e^{i\omega x}}{x^4+1}\,dx$

- $\displaystyle\int_{\gamma R} f(z)\,dz \xrightarrow[R\to+\infty]{} 0$: VERLO EN DETALLE. EL PUNTO CLAVE ES:

$i\omega z = i\omega\,[R\cos(\theta) + i R\,\mathrm{sen}(\theta)]$

$= -\underbrace{R\omega}_{>0\ \checkmark}\,\mathrm{sen}(\theta) + i\omega R\cos(\theta)$

$\therefore\ \forall \omega > 0$:

$$\vec{f}(\omega) = \int_{-\infty}^{+\infty} \frac{x\, e^{i\omega x}}{x^4+1}\,dx = \frac{\pi\sqrt{2}}{2}\, e^{-\frac{\omega}{\sqrt{2}}} \left[\cos\left(\frac{\omega}{\sqrt{2}}\right) + \mathrm{sen}\left(\frac{\omega}{\sqrt{2}}\right)\right]$$

AHORA:

$$\vec{f}(\omega) = \underbrace{\int_{-\infty}^{+\infty} \frac{x\cos(\omega x)}{x^4+1}\,dx}_{=0} + i \underbrace{\int_{-\infty}^{+\infty} \frac{x\,\mathrm{sen}(\omega x)}{x^4+1}\,dx}_{\text{IMPAR RESPECTO DE } \omega}$$

$\therefore\ \forall \omega < 0$: $-\omega > 0$

$$\vec{f}(\omega) = -\vec{f}(-\omega) = -\frac{\pi\sqrt{2}}{2}\, e^{\frac{\omega}{\sqrt{2}}} \left[\cos\left(\frac{-\omega}{\sqrt{2}}\right) + \mathrm{sen}\left(\frac{-\omega}{\sqrt{2}}\right)\right] =$$

$\vec{f}$ ES IMPAR

$$= -\frac{\pi\sqrt{2}}{2}\, e^{\frac{\omega}{\sqrt{2}}} \left[\cos\left(\frac{\omega}{\sqrt{2}}\right) - \mathrm{sen}\left(\frac{\omega}{\sqrt{2}}\right)\right] =$$

$$= \frac{\pi\sqrt{2}}{2}\, e^{-\frac{|\omega|}{2}} \left[-\cos\left(\frac{\omega}{\sqrt{2}}\right) + \mathrm{sen}\left(\frac{\omega}{\sqrt{2}}\right)\right]$$

$$\therefore \forall \omega \neq 0: \ \vec{f}(\omega) = \begin{cases} \frac{\pi}{\sqrt{2}} e^{-\frac{|\omega|}{\sqrt{2}}} \left[\cos\left(\frac{\omega}{\sqrt{2}}\right) + \operatorname{sen}\left(\frac{\omega}{\sqrt{2}}\right) \right] & \text{si } \omega > 0 \\ \frac{\pi}{\sqrt{2}} e^{-\frac{|\omega|}{\sqrt{2}}} \left[-\cos\left(\frac{\omega}{\sqrt{2}}\right) + \operatorname{sen}\left(\frac{\omega}{\sqrt{2}}\right) \right] & \text{si } \omega < 0 \end{cases}$$

FINALMENTE:

$$\vec{f}(0) = \underbrace{\int_{-\infty}^{+\infty} \frac{x}{x^4+1} \, dx}_{\text{Impar}} = 0 = \frac{1}{2} \Big[\underbrace{\vec{f}(0^+)}_{\frac{\pi}{\sqrt{2}}} + \underbrace{\vec{f}(0^-)}_{-\frac{\pi}{\sqrt{2}}} \Big] \ \checkmark$$

OBS1. PARA EL CÁLCULO DE $\vec{f}(\omega)$ CON $\omega < 0$ SE PUEDE UTILIZAR, OBVIAMENTE, EL CIRCUITO SIMÉTRICO (RESPECTO DE $\mathbb{R}$) DEL UTILIZADO EN PAG. 5.

OBS 2. $\forall \omega \in \mathbb{R}$:

$$\boxed{\begin{gathered} \vec{f}(\omega) = \frac{\pi}{\sqrt{2}} e^{-\frac{|\omega|}{\sqrt{2}}} \left[\operatorname{SIGN}(\omega) \cos\left(\frac{\omega}{\sqrt{2}}\right) + \operatorname{sen}\left(\frac{\omega}{\sqrt{2}}\right) \right] \\ \text{DONDE } \operatorname{SIGN}(\omega) = \begin{cases} -1 & \text{si } \omega < 0 \\ 0 & \text{si } \omega = 0 \\ 1 & \text{si } \omega > 0 \end{cases} \end{gathered}}$$

4| (I) $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \qquad x>0,\ t>0$

(II) $u(x,0) = 0 \qquad x>0$

(III) $\frac{\partial u}{\partial t}(x,0) = f(x) \qquad x>0$

(IV) $u(0,t) = 0 \qquad t>0$

PARA CUALQUIER PAR DE FUNCIONES $\alpha: \mathbb{R} \to \mathbb{R}$ Y $\beta: \mathbb{R} \to \mathbb{R}$ DE CLASE C^2, $u(x,t) = \alpha(x+t) + \beta(x-t)$ ES SOLUCIÓN DE (I).

AHORA:

(II) $u(x,0) = 0 \iff \alpha(x) + \beta(x) = 0$: $\beta = -\alpha$:

$$u(x,t) = \alpha(x+t) - \alpha(x-t) \qquad (1)$$

(III) $\frac{\partial}{\partial t} u(x,0) = f(x) \iff \alpha'(x) + \alpha'(x) = f(x)$: $\alpha(x) = \frac{1}{2}\int_0^x f(\theta)\,d\theta$:

$$u(x,t) = \frac{1}{2}\int_0^{x+t} f(\theta)\,d\theta - \frac{1}{2}\int_0^{x-t} \tilde{f}(\theta)\,d\theta \qquad (2)$$

$\uparrow$ $\tilde{f}$: EXTENSIÓN DE f A $\mathbb{R}$

(IV) $u(0,t) = 0 \iff \frac{1}{2}\int_0^{t} f(\theta)\,d\theta - \frac{1}{2}\int_0^{-t} \tilde{f}(\theta)\,d\theta \underset{\theta = -\tau}{=}$

$$= \frac{1}{2}\int_0^{t} f(\theta)\,d\theta - \frac{1}{2}\int_0^{t} \tilde{f}(-\tau)(-d\tau) =$$

$$= \frac{1}{2}\int_0^{t} f(\theta)\,d\theta + \frac{1}{2}\int_0^{t} \tilde{f}(-\tau)\,d\tau =$$

$$= \frac{1}{2}\int_0^{t} \{f(\theta) + \tilde{f}(-\theta)\}\,d\theta = 0$$

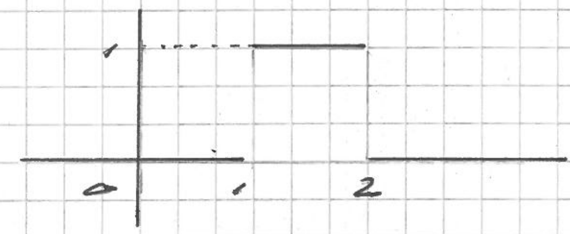
$\leftarrow \tilde{f}(-\theta) = -f(\theta)$: $\tilde{f}$: EXTENSIÓN IMPAR

RESPUESTA: $u(x,t) = \frac{1}{2}\int_0^{x+t} \tilde{f}(\theta)\,d\theta - \frac{1}{2}\int_0^{x-t} \tilde{f}(\theta)\,d\theta$

: $\tilde{f}$ = EXTENSION IMPAR DE f.

: f DE CLASE C^1

5) $$y'(t) + \underbrace{\int_0^t y(u)\, H(t-u)\, du}_{(y * H)(t)} = H(t-1) - H(t-2)$$



$\downarrow \mathcal{L}$

$$sY(s) - \overbrace{y(0+)}^{=0} + Y(s)\cdot\frac{1}{s} = \int_1^2 e^{-st}\, dt = \frac{e^{-2s} - e^{-s}}{-s} = \frac{e^{-s} - e^{-2s}}{s}$$

$$\left(s + \frac{1}{s}\right) Y(s) = \frac{e^{-s} - e^{-2s}}{s} \quad \times s$$

$$(s^2+1)\, Y(s) = e^{-s} - e^{-2s}$$

$$Y(s) = \frac{1}{s^2+1} e^{-s} - \frac{1}{s^2+1} e^{-2s}$$

$$\boxed{y(t) = \sin(t-1)\, H(t-1) - \sin(t-2)\, H(t-2)}$$