Universidad de Buenos Aires

Estabilidad I

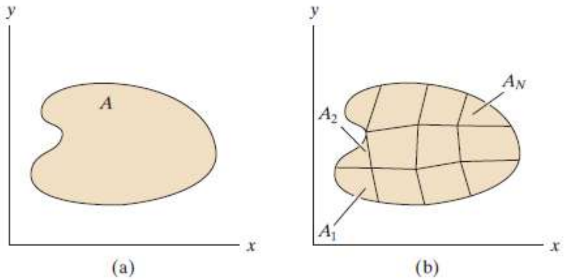
Baricentros- Momentos de Inercia

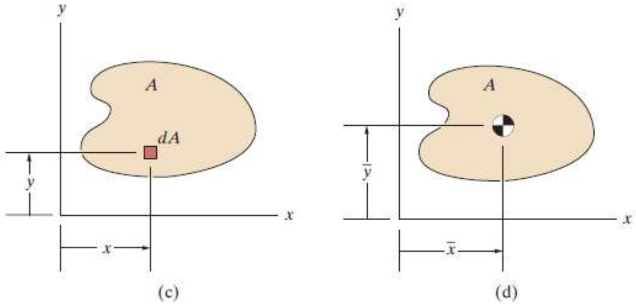
Estática de Bedford Fowler

Baricentro en el plano

Posicion Media de Puntos Materiales $\bar{x} = \frac{x_1 + x_2 + \cdots + x_N}{N} = \frac{\sum_i x_i}{N}, \qquad \bar{y} = \frac{\sum_i y_i}{N}.$

Baricentro de Areas

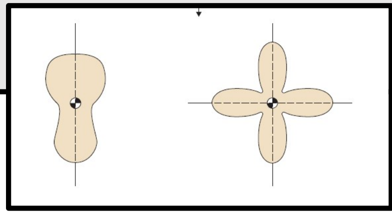




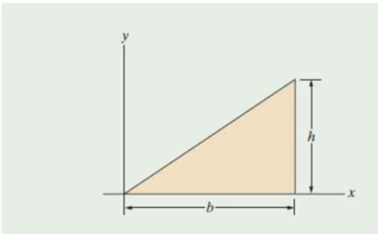
$$\bar{x} = \frac{\sum_i x_i A_i}{\sum_i A_i}, \qquad \bar{y} = \frac{\sum_i y_i A_i}{\sum_i A_i}.$$

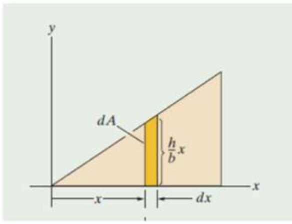
$$\bar{x} = \frac{\int_A x\, dA}{\int_A dA} \qquad \bar{y} = \frac{\int_A y\, dA}{\int_A dA},$$

Baricentro de superficies plana



Determinar el Baricentro del area triangular

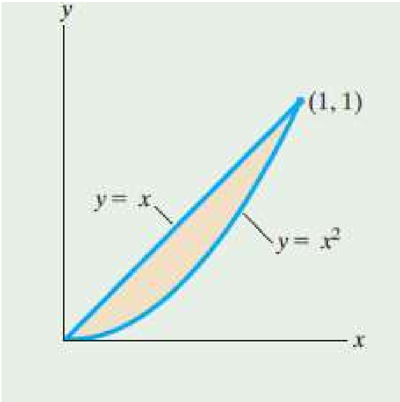


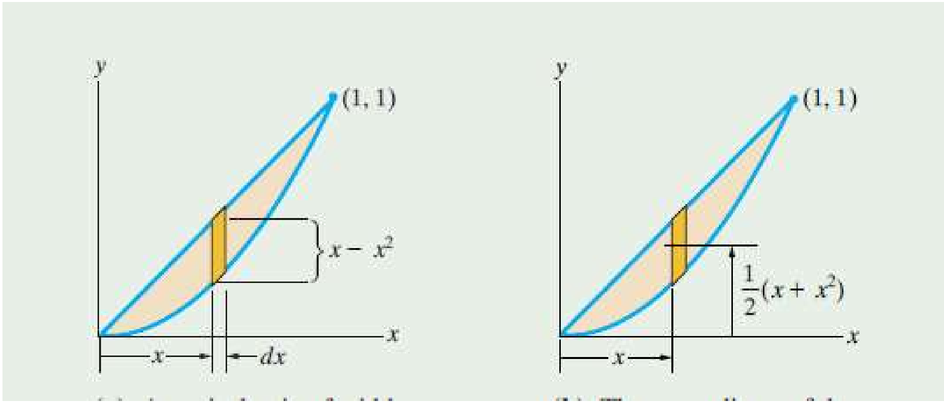


$$\bar{x} = \frac{\int_A x\,dA}{\int_A dA} = \frac{\int_0^b x\left(\frac{h}{b}x\,dx\right)}{\int_0^b \frac{h}{b}x\,dx} = \frac{\frac{h}{b}\left[\frac{x^3}{3}\right]_0^b}{\frac{h}{b}\left[\frac{x^2}{2}\right]_0^b} = \frac{2}{3}b.$$

$$\bar{y} = \frac{\int_A y\,dA}{\int_A dA} = \frac{\int_0^b \frac{1}{2}\left(\frac{h}{b}x\right)\left(\frac{h}{b}x\,dx\right)}{\int_0^b \frac{h}{b}x\,dx} = \frac{\frac{1}{2}\left(\frac{h}{b}\right)^2\left[\frac{x^3}{3}\right]_0^b}{\frac{h}{b}\left[\frac{x^2}{2}\right]_0^b} = \frac{1}{3}h.$$

Calcular Baricentro

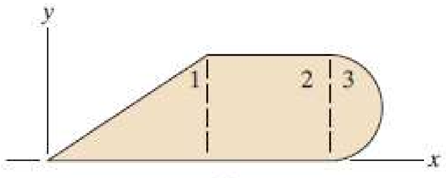




$$\bar{x} = \frac{\displaystyle\int_A x\,dA}{\displaystyle\int_A dA} = \frac{\displaystyle\int_0^1 x(x - x^2)\,dx}{\displaystyle\int_0^1 (x - x^2)\,dx} = \frac{\left[\dfrac{x^3}{3} - \dfrac{x^4}{4}\right]_0^1}{\left[\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_0^1} = \frac{1}{2}.$$

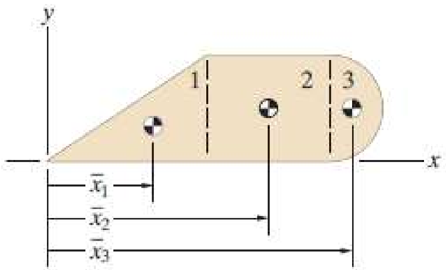
$$\bar{y} = \frac{\displaystyle\int_A y\,dA}{\displaystyle\int_A dA} = \frac{\displaystyle\int_0^1 \left[\frac{1}{2}(x + x^2)\right](x - x^2)\,dx}{\displaystyle\int_0^1 (x - x^2)\,dx} = \frac{\dfrac{1}{2}\left[\dfrac{x^3}{3} - \dfrac{x^5}{5}\right]_0^1}{\left[\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_0^1} = \frac{2}{5}.$$

Areas Compuestas



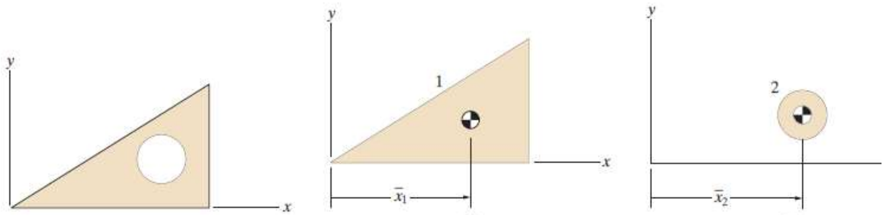
(a)

$$\bar{x} = \frac{\int_A x\,dA}{\int_A dA} = \frac{\int_{A_1} x\,dA + \int_{A_2} x\,dA + \int_{A_3} x\,dA}{\int_{A_1} dA + \int_{A_2} dA + \int_{A_3} dA}.$$



(b)

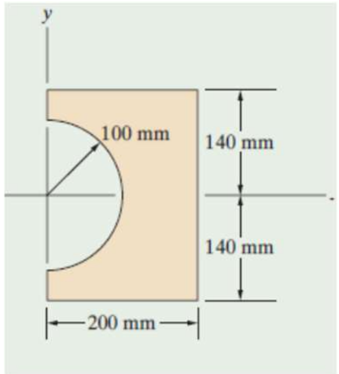
$$\bar{x}_1 = \frac{\int_{A_1} x\,dA}{\int_{A_1} dA}, \qquad \int_{A_1} x\,dA = \bar{x}_1 A_1. \qquad \bar{x} = \frac{\bar{x}_1 A_1 + \bar{x}_2 A_2 + \bar{x}_3 A_3}{A_1 + A_2 + A_3}.$$

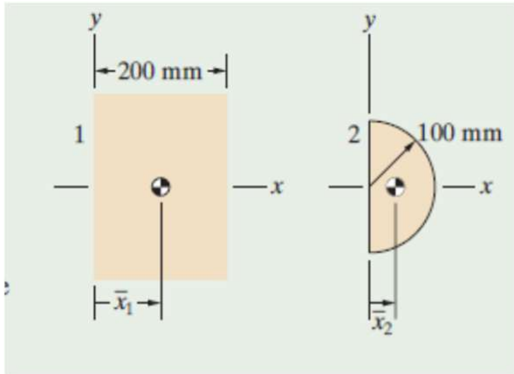


(a) (b) (c)

$$\bar{x} = \frac{\int_{A_1} x\,dA - \int_{A_2} x\,dA}{\int_{A_1} dA - \int_{A_2} dA} = \frac{\bar{x}_1 A_1 - \bar{x}_2 A_2}{A_1 - A_2}.$$

Ejercicio



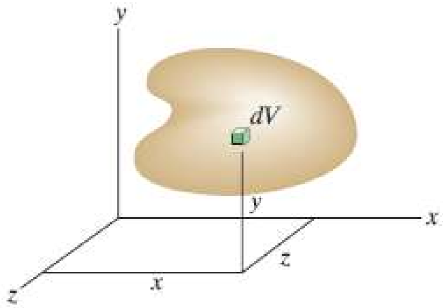


$$\bar{x}_2 = \frac{4R}{3\pi} = \frac{4(100)}{3\pi} \text{ mm.}$$

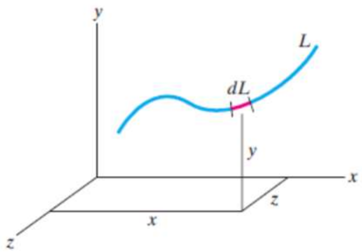
	$\bar{x}_i$ (mm)	A_i (mm^2)	$\bar{x}_i A_i$ (mm^3)
Part 1 (rectangle)	100	(200)(280)	(100)[(200)(280)]
Part 2 (cutout)	$\frac{4(100)}{3\pi}$	$-\frac{1}{2}\pi(100)^2$	$-\frac{4(100)}{3\pi}\left[\frac{1}{2}\pi(100)^2\right]$

$$\bar{x} = \frac{\bar{x}_1 A_1 + \bar{x}_2 A_2}{A_1 + A_2} = \frac{(100)[(200)(280)] - \frac{4(100)}{3\pi}\left[\frac{1}{2}\pi(100)^2\right]}{(200)(280) - \frac{1}{2}\pi(100)^2} = 122 \text{ mm}$$

Baricentro para volumenes y linea en el espacio

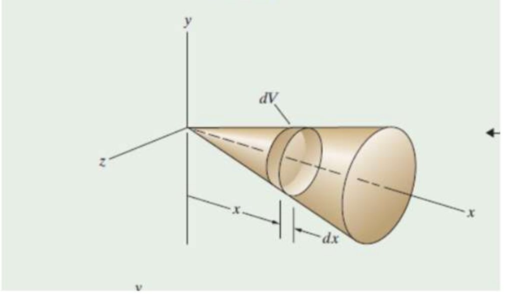


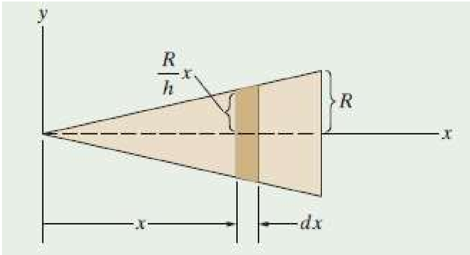
$$\bar{x} = \frac{\int_V x\, dV}{\int_V dV}, \qquad \bar{y} = \frac{\int_V y\, dV}{\int_V dV}, \qquad \bar{z} = \frac{\int_V z\, dV}{\int_V dV}.$$



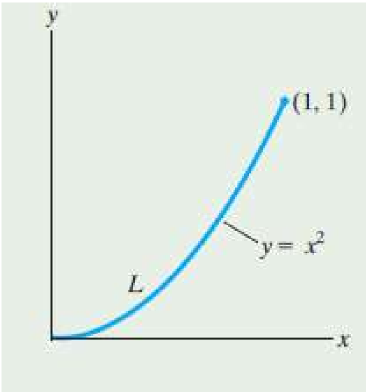
$$\bar{x} = \frac{\int_L x\, dL}{\int_L dL}, \qquad \bar{y} = \frac{\int_L y\, dL}{\int_L dL}, \qquad \bar{z} = \frac{\int_L z\, dL}{\int_L dL},$$

Solution



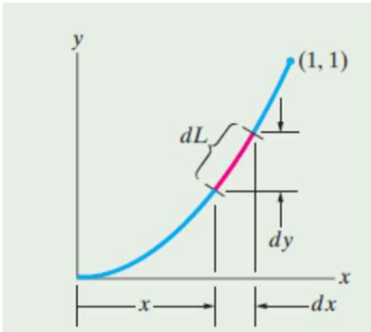


$$\bar{x} = \frac{\int_V x\,dV}{\int_V dV} = \frac{\int_0^h x\pi\left[\left(\frac{R}{h}\right)x\right]^2}{\int_0^h \pi\left[\left(\frac{R}{h}\right)x\right]^2 dx} = \frac{3}{4}h.$$

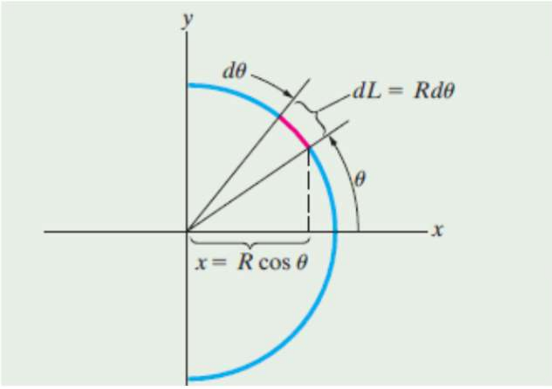


$$dL = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

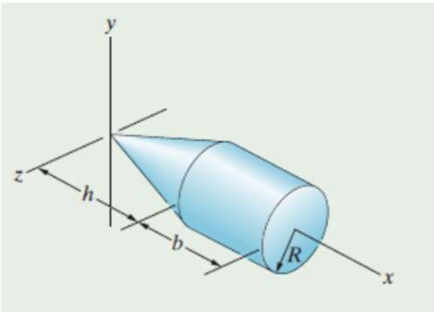
$$dL = \sqrt{1 + 4x^2}\,dx.$$

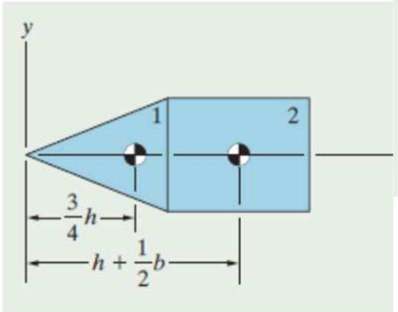


$$\bar{x} = \frac{\int_L x\,dL}{\int_L dL} = \frac{\int_0^1 x\sqrt{1 + 4x^2}\,dx}{\int_0^1 \sqrt{1 + 4x^2}\,dx} = 0.574.$$



$$\bar{x} = \frac{\int_L x\,dL}{\int_L dL} = \frac{\int_{-\pi/2}^{\pi/2} (R\cos\theta) R\,d\theta}{\int_{-\pi/2}^{\pi/2} R\,d\theta} = \frac{R^2[\sin\theta]_{-\pi/2}^{\pi/2}}{R[\theta]_{-\pi/2}^{\pi/2}} = \frac{2R}{\pi}.$$



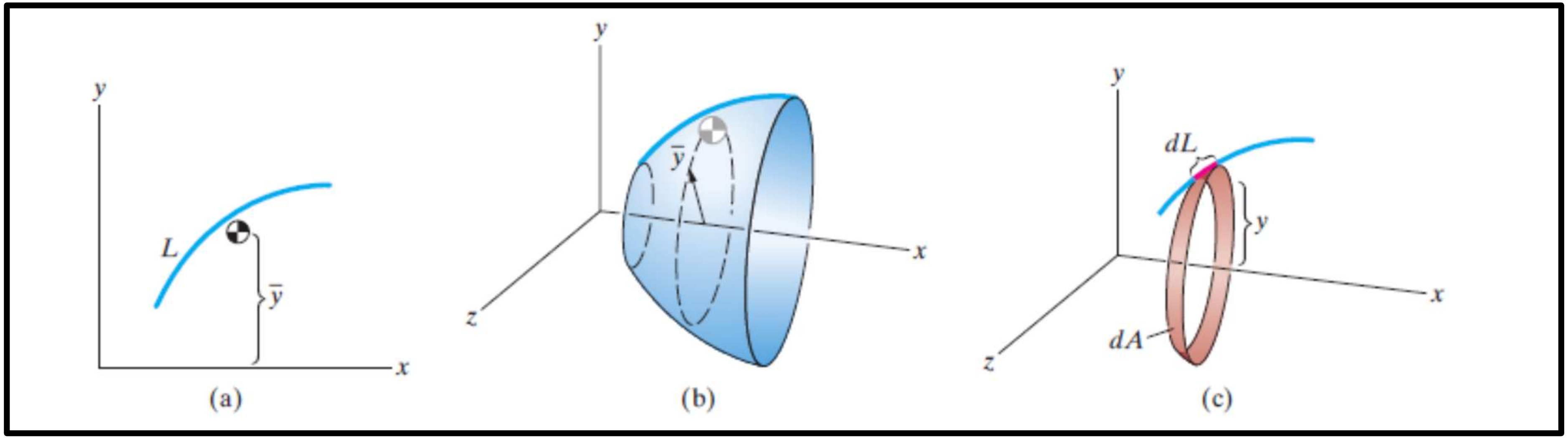


	$\bar{x}_i$	V_i	$\bar{x}_i V_i$
Part 1 (cone)	$\frac{3}{4}h$	$\frac{1}{3}\pi R^2 h$	$\left(\frac{3}{4}h\right)\left(\frac{1}{3}\pi R^2 h\right)$
Part 2 (cylinder)	$h + \frac{1}{2}b$	$\pi R^2 b$	$\left(h + \frac{1}{2}b\right)(\pi R^2 b)$

$$\bar{x} = \frac{\bar{x}_1 V_1 + \bar{x}_2 V_2}{V_1 + V_2}.$$

$$= \frac{\left(\frac{3}{4}h\right)\left(\frac{1}{3}\pi R^2 h\right) + \left(h + \frac{1}{2}b\right)\left(\pi R^2 b\right)}{\frac{1}{3}\pi R^2 h + \pi R^2 b}.$$

***Teorema de Pappus I:* El area de la sup de revolucion es igual al producto de la distancia que el baricentro de la linea recorre por la longitud de la linea.**



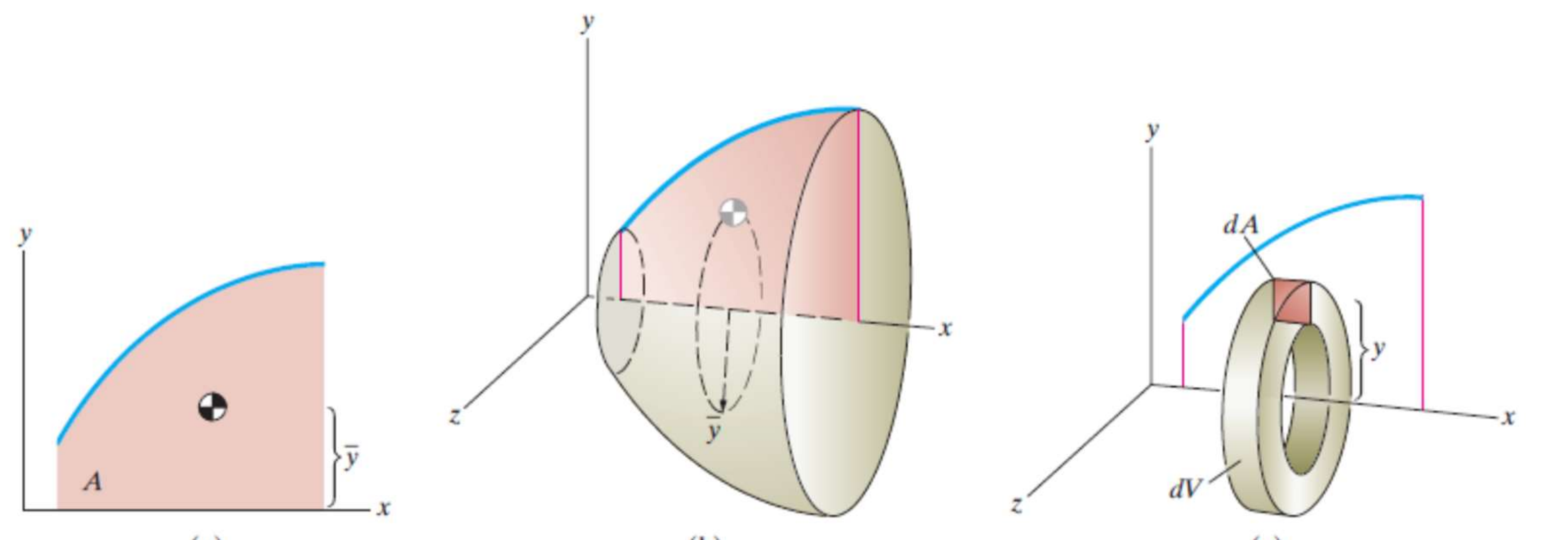
$$A = 2\pi \bar{y} L.$$

$$A = 2\pi \int_L y \, dL.$$

$$\bar{y} = \frac{\int_L y \, dL}{\int_L dL},$$

$$\int_L y \, dL = \bar{y} L.$$

Teorema de Pappus II: El vol. de revolucion es igual al producto de la distancia que recorre el baricentro del area por la magnitud del area

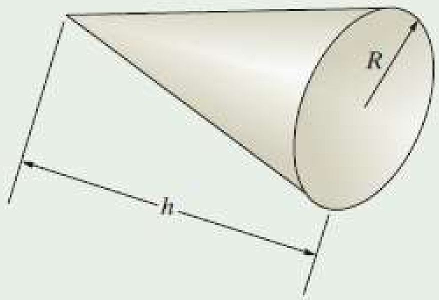


(a) (b) (c)

$$V = 2\pi \bar{y} A. \qquad V = 2\pi \int_A y\, dA. \qquad \bar{y} = \frac{\int_A y\, dA}{\int_A dA}, \qquad \int_A y\, dA = \bar{y} A.$$

Ejemplos

Use the first Pappus–Guldinus theorem to determine the surface area of the cone.



Strategy

We can generate the curved surface of the cone by revolving a straight line about an axis. Because the location of the centroid of the straight line is known, we can use the first Pappus–Guldinus theorem to determine the area of the curved surface.

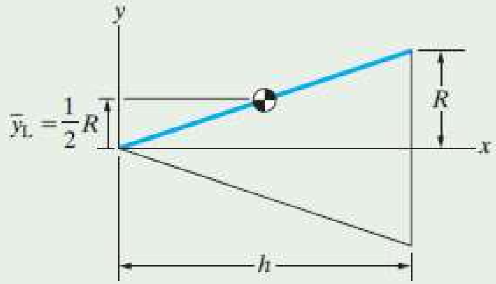
Solution

Revolving this straight line about the x axis generates the curved surface of the cone. The y coordinate of the centroid of the line is shown.

The length of the line is $L = \sqrt{h^2 + R^2}$.
The area of the curved surface is

$$A = 2\pi \bar{y}_L L = \pi R \sqrt{h^2 + R^2}.$$

Adding the area of the base, the total surface area of the cone is $\pi R \sqrt{h^2 + R^2} + \pi R^2$.



Practice Problem Use the second Pappus–Guldinus theorem to determine the volume of the cone.

Answer: $V = \frac{1}{3}\pi h R^2$.

Ejemplos

Example 7.15 Determining a Centroid with a Pappus–Guldinus Theorem (▶ *Related Problem 7.88*)

The circumference of a sphere of radius R is $2\pi R$ and its surface area is $4\pi R^2$. Use this information to determine the centroid of a semicircular line.

Strategy

Revolving a semicircular line about an axis generates a spherical area. Knowing the area, we can use the first Pappus–Guldinus theorem to determine the centroid of the generating line.

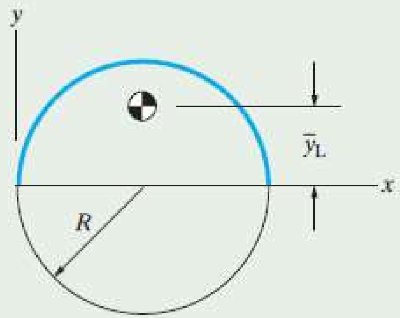
Solution

The length of the semicircular line is $L = \pi R$, and $\bar{y}_L$ is the y coordinate of its centroid. Rotating the line about the x axis generates the surface of a sphere. The first Pappus–Guldinus theorem states that the surface area of the sphere is

$$(2\pi\bar{y}_L)L = 2\pi^2 R\bar{y}_L.$$

By equating this expression to the given surface area $4\pi R^2$, we obtain $\bar{y}_L$:

$$\bar{y}_L = \frac{2R}{\pi}.$$



Revolving a semicircular line about the x axis.

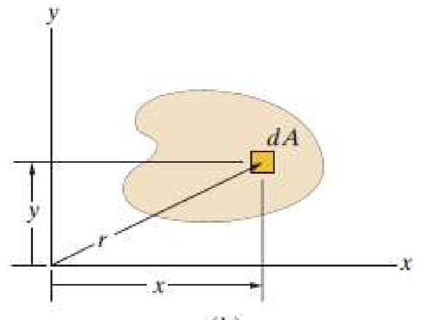
Critical Thinking

If you can obtain a result by using the Pappus–Guldinus theorems, you will often save time and effort in comparison with other approaches. Compare this example with Example 7.10, in which we used integration to determine the centroid of a semicircular line.

Momentos de Inercia

$$I_x = \int_A y^2\, dA, \qquad I_x = k_x^2 A.$$

$$I_y = \int_A x^2\, dA, \qquad I_y = k_y^2 A.$$



(b)

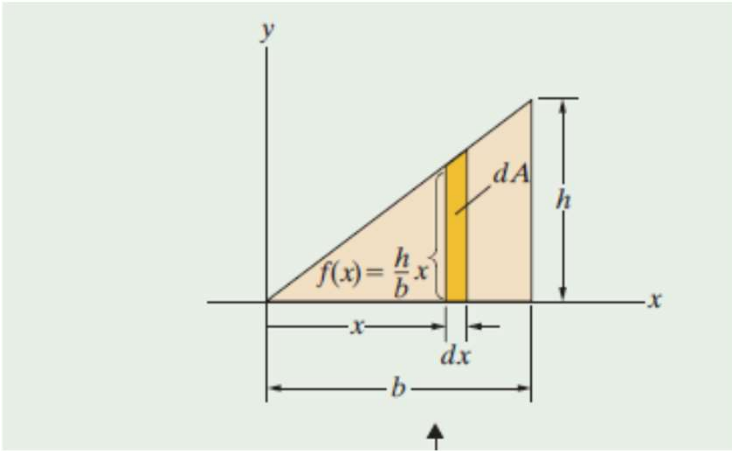
$$I_{xy} = \int_A xy\, dA.$$

$$J_O = \int_A r^2\, dA, \qquad J_O = k_O^2 A. \qquad J_O = \int_A r^2\, dA = \int_A (y^2 + x^2)\, dA = I_x + I_y.$$

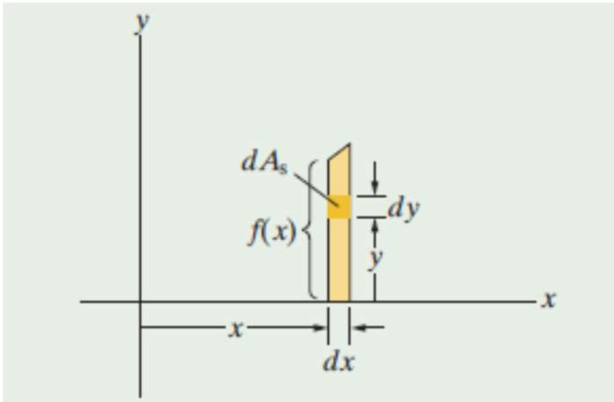
$$k_O^2 = k_x^2 + k_y^2.$$

Ejemplo

$$
\begin{aligned}
I_y &= \int_A x^2\, dA \\
&= \int_0^b x^2 f(x)\, dx \\
&= \int_0^b x^2 \left(\frac{h}{b}\right) x\, dx \\
&= \frac{1}{4} h b^3.
\end{aligned}
$$

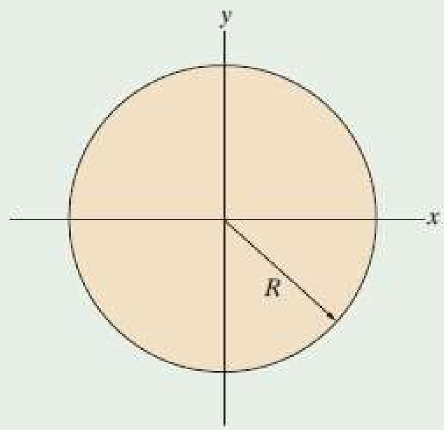


$$
\begin{aligned}
(I_x)_{\text{strip}} &= \int_{\text{strip}} y^2\, dA_s \\
&= \int_0^{f(x)} (y^2 dx)\, dy \\
&= \frac{1}{3} [\, f(x)]^3 dx.
\end{aligned}
$$



Example 8.2 Moments of Inertia of a Circular Area (▶ *Related Problem 8.21*)

Determine the moments of inertia and radii of gyration of the circular area.



Strategy

We will first determine the polar moment of inertia J_O by integrating in terms of polar coordinates. We know from the symmetry of the area that $I_x = I_y$, and since $I_x + I_y = J_O$, the moments of inertia I_x and I_y are each equal to $\frac{1}{2}J_O$. We also know from the symmetry of the area that $I_{xy} = 0$.

Solution

By letting r change by an amount dr, we obtain an annular element of area $dA = 2\pi r\,dr$ (Fig. a). The polar moment of inertia is

$$J_O = \int_A r^2\,dA = \int_0^R 2\pi r^3\,dr = 2\pi\left[\frac{r^4}{4}\right]_0^R = \frac{1}{2}\pi R^4,$$

and the radius of gyration about O is

$$k_O = \sqrt{\frac{J_O}{A}} = \sqrt{\frac{(1/2)\pi R^4}{\pi R^2}} = \frac{1}{\sqrt{2}}R.$$

The moments of inertia about the x and y axes are

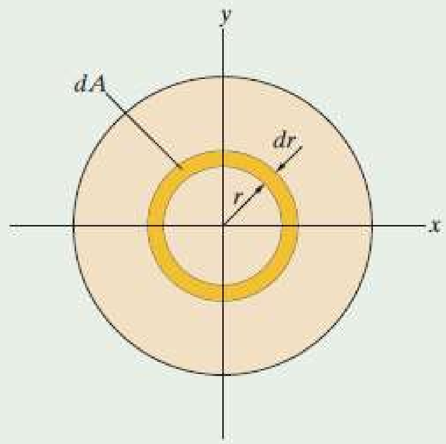
$$I_x = I_y = \frac{1}{2}J_O = \frac{1}{4}\pi R^4,$$

and the radii of gyration about the x and y axes are

$$k_x = k_y = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{(1/4)\pi R^4}{\pi R^2}} = \frac{1}{2}R.$$

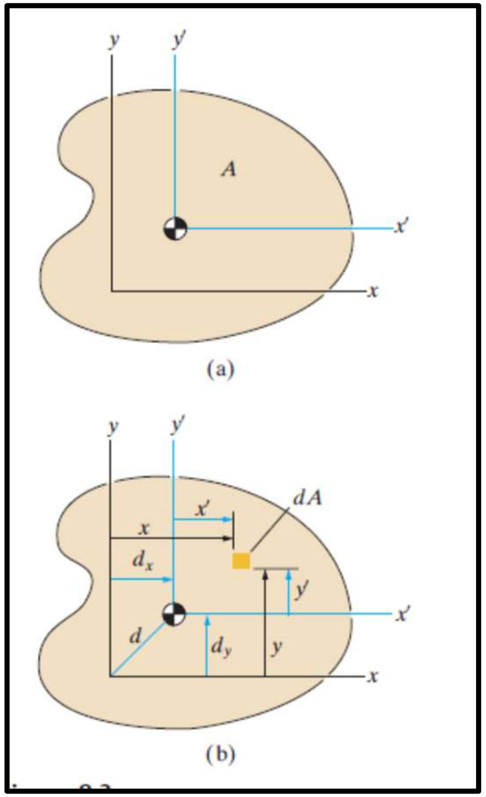
The product of inertia is zero:

$$I_{xy} = 0.$$



(a) An annular element dA.

Teorema de los Ejes Paralelos (Steiner)

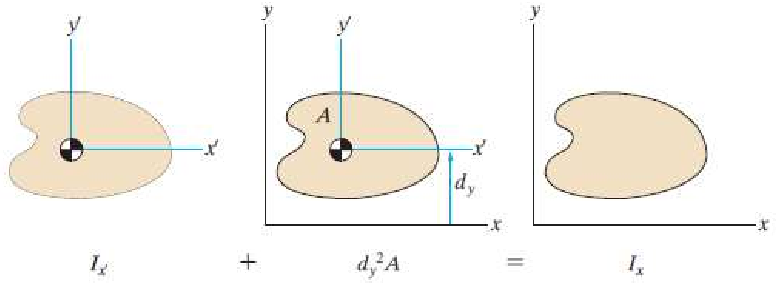


Se conocen los momentos de inercia de un area A en terminos de un Sistema de ejes coordenados con su origen en el baricentro (x'; y').

$$\overline{x}' = \frac{\int_A x' dA}{\int_A dA}, \qquad \overline{y}' = \frac{\int_A y' dA}{\int_A dA}. \qquad \int_A x' dA = 0, \quad \int_A y' dA = 0.$$

$$I_x = \int_A y^2 \, dA, \qquad I_x = \int_A (y' + d_y)^2 \, dA = \int_A (y')^2 \, dA + 2d_y \int_A y' dA + d_y^2 \int_A dA.$$

$$I_x = I_{x'} + d_y^2 A.$$



Teorema de los Ejes Paralelos (Steiner)

$$I_y = I_{y'} + d_x^2 A.$$

$$I_{xy} = \int_A xy\, dA = \int_A (x' + d_x)(y' + d_y)\, dA$$

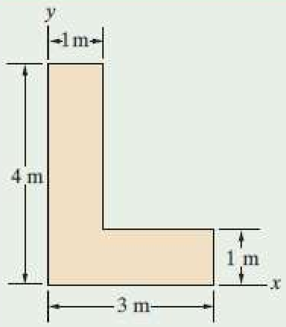
$$= \int_A x'y'dA + d_y \int_A x'dA + d_x \int_A y'dA + d_x d_y \int_A dA.$$

$$I_{xy} = I_{x'y'} + d_x d_y A.$$

$$J_O = J'_O + (d_x^2 + d_y^2)A = J'_O + d^2 A,$$

Ejemplo

Active Example 8.3 Moments of Inertia of a Composite Area (► *Related Problem 8.27*)

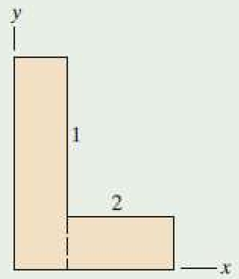


Determine I_x for the composite area.

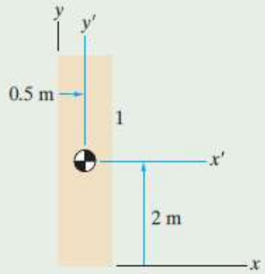
Strategy

We can divide this area into two rectangles. We must use the parallel-axis theorems to determine I_x for each rectangle in terms of the xy coordinate system. The values can be summed to determine I_x for the composite area.

Solution



Divide the composite area into two rectangles.



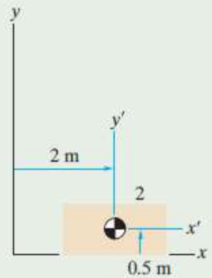
From Appendix B, the moment of inertia of area 1 about the x' axis is

$$(I_{x'})_1 = \frac{1}{12}(1\ \text{m})(4\ \text{m})^3 = 5.33\ \text{m}^4.$$

Applying the parallel-axis theorem, the moment of inertia of area 1 about the x axis is

$$(I_x)_1 = 5.33\ \text{m}^4 + (2\ \text{m})^2(1\ \text{m})(4\ \text{m}) = 21.3\ \text{m}^4.$$

Apply Eq. (8.10) to area 12.



The moment of inertia of area 2 about the x' axis is

$$(I_{x'})_2 = \frac{1}{12}(2\ \text{m})(1\ \text{m})^3 = 0.167\ \text{m}^4.$$

Applying the parallel-axis theorem, the moment of inertia of area 2 about the x axis is

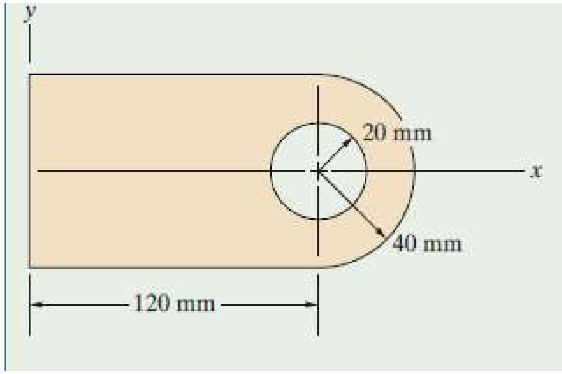
$$(I_x)_2 = 0.167\ \text{m}^4 + (0.5\ \text{m})^2(2\ \text{m})(1\ \text{m}) = 0.667\ \text{m}^4.$$

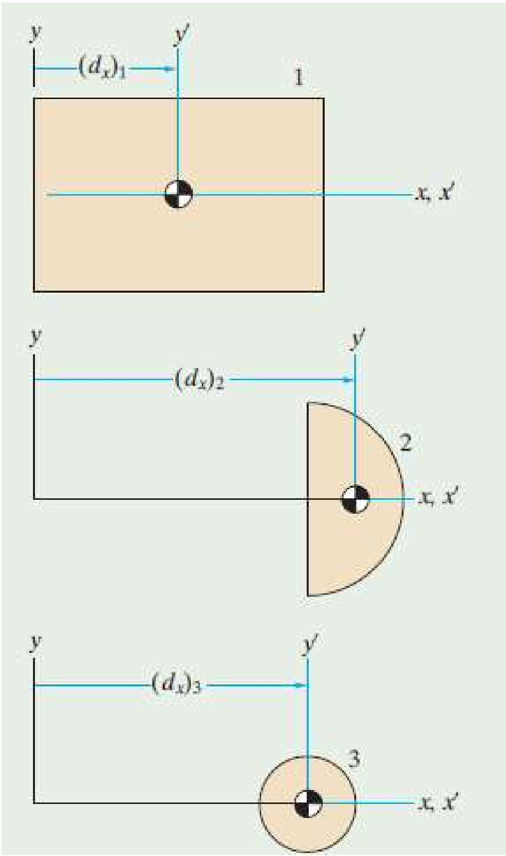
Apply Eq. (8.10) to area 2.

The moment of inertia of the composite area about the x axis is

$$\begin{aligned} I_x &= (I_x)_1 + (I_x)_2 \\ &= 21.3\ \text{m}^4 + 0.667\ \text{m}^4 \\ &= 22.0\ \text{m}^4. \end{aligned}$$

Sum the values for the parts.





	d_x (mm)	A (mm^2)	$I_{y'}$ (mm^4)	$I_y = I_{y'} + d_x^2 A$ (mm^4)
Part 1	60	(120)(80)	$\frac{1}{12}(80)(120)^3$	4.608×10^7
Part 2	$120 + \frac{4(40)}{3\pi}$	$\frac{1}{2}\pi(40)^2$	$\left(\frac{\pi}{8} - \frac{8}{9\pi}\right)(40)^4$	4.744×10^7
Part 3	120	$\pi(20)^2$	$\frac{1}{4}\pi(20)^4$	1.822×10^7

Sum the Results The moment of inertia of the composite area about the y axis is

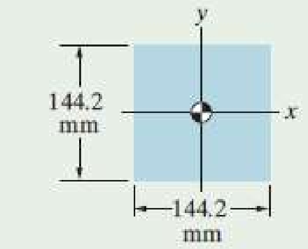
$$I_y = (I_y)_1 + (I_y)_2 - (I_y)_3 = (4.608 + 4.744 - 1.822) \times 10^7 \text{ mm}^4$$
$$= 7.530 \times 10^7 \text{ mm}^4.$$

The total area is

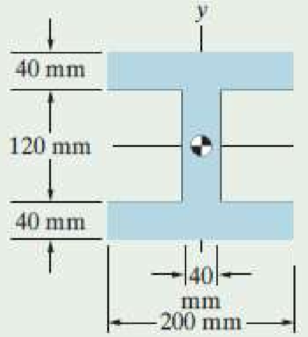
$$A = A_1 + A_2 - A_3 = (120 \text{ mm})(80 \text{ mm}) + \frac{1}{2}\pi(40 \text{ mm})^2 - \pi(20 \text{ mm})^2$$
$$= 1.086 \times 10^4 \text{ mm}^2,$$

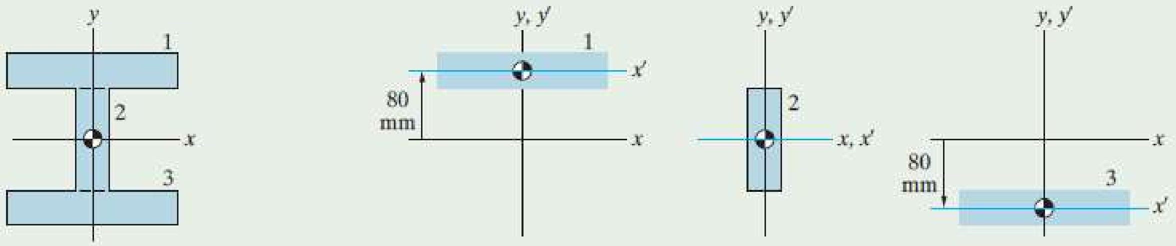
so the radius of gyration about the y axis is

$$k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{7.530 \times 10^7 \text{ mm}^4}{1.086 \times 10^4 \text{ mm}^2}} = 83.3 \text{ mm}.$$



$$I_x = \frac{1}{12}(144.2\text{ mm})(144.2\text{ mm})^3 = 3.60 \times 10^7\text{ mm}^4.$$





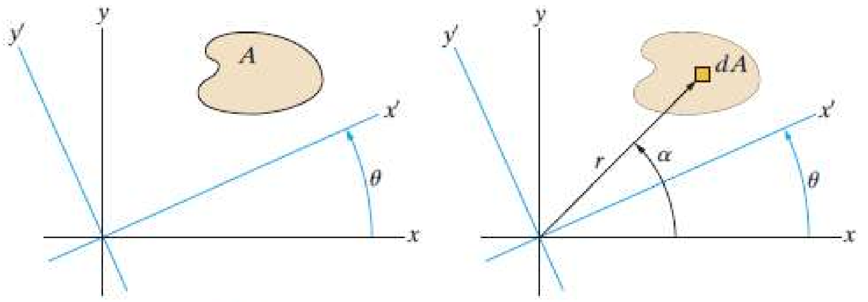
Determining the moments of inertia of the parts about the *x* axis.

	d_y (mm)	A (mm^2)	$I_{x'}$ (mm^4)	$I_x = I_{x'} + d_y^2 A$ (mm^4)
Part 1	80	(200)(40)	$\frac{1}{12}(200)(40)^3$	5.23×10^7
Part 2	0	(40)(120)	$\frac{1}{12}(40)(120)^3$	0.58×10^7
Part 3	−80	(200)(40)	$\frac{1}{12}(200)(40)^3$	5.23×10^7

$$I_x = (I_x)_1 + (I_x)_2 + (I_x)_3 = (5.23 + 0.58 + 5.23) \times 10^7\text{ mm}^4$$

$$= 11.03 \times 10^7\text{ mm}^4.$$

Ejes Girados



(a) (b)

$$x' = r\cos(\alpha - \theta) = r(\cos\alpha\cos\theta + \sin\alpha\sin\theta),$$
$$y' = r\sin(\alpha - \theta) = r(\sin\alpha\cos\theta - \cos\alpha\sin\theta).$$

$$x' = x\cos\theta + y\sin\theta,$$
$$y' = -x\sin\theta + y\cos\theta.$$

Moment of Inertia About the x' Axis

$$I_{x'} = \int_A (y')^2\,dA = \int_A (-x\sin\theta + y\cos\theta)^2\,dA$$
$$= \cos^2\theta\int_A y^2\,dA - 2\sin\theta\cos\theta\int_A xy\,dA + \sin^2\theta\int_A x^2\,dA.$$

From this equation we obtain

$$I_{x'} = I_x\cos^2\theta - 2I_{xy}\sin\theta\cos\theta + I_y\sin^2\theta. \qquad (8.20)$$

Ejes Girados

Moment of Inertia About the y' Axis

$$I_{y'} = \int_A (x')^2 \, dA = \int_A (x \cos\theta + y \sin\theta)^2 \, dA$$

$$= \sin^2\theta \int_A y^2 \, dA + 2 \sin\theta \cos\theta \int_A xy \, dA + \cos^2\theta \int_A x^2 \, dA.$$

This equation gives us the result

$$I_{y'} = I_x \sin^2\theta + 2I_{xy} \sin\theta \cos\theta + I_y \cos^2\theta. \qquad (8.21)$$

Product of Inertia In terms of the $x'y'$ coordinate system, the product of inertia of A is

$$I_{x'y'} = (I_x - I_y) \sin\theta \cos\theta + (\cos^2\theta - \sin^2\theta) I_{xy}. \qquad (8.22)$$

Polar Moment of Inertia From Eqs. (8.20) and (8.21), the polar moment of inertia in terms of the $x'y'$ coordinate system is

$$J'_O = I_{x'} + I_{y'} = I_x + I_y = J_O.$$

Thus the value of the polar moment of inertia is unchanged by a rotation of the coordinate system.

Ejes Girados Ejes Principales

$$\sin 2\theta = 2\sin\theta\cos\theta,$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1.$$

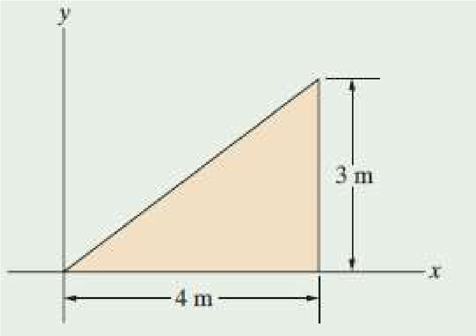
$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta,$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta,$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta.$$

$$\tan 2\theta_p = \frac{2I_{xy}}{I_y - I_x}.$$

$$\frac{d^2 I_{x'}}{d(2\theta)^2} = -\frac{d^2 I_{y'}}{d(2\theta)^2},$$



$$I_x = \frac{1}{12}(4\text{ m})(3\text{ m})^3 = 9\text{ m}^4,$$

$$I_y = \frac{1}{4}(4\text{ m})^3(3\text{ m}) = 48\text{ m}^4,$$

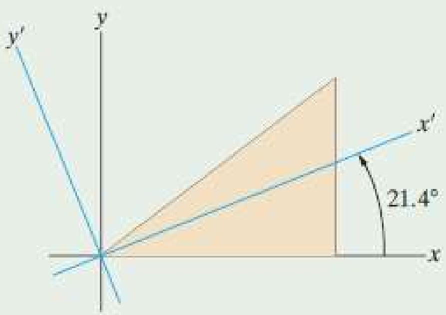
$$I_{xy} = \frac{1}{8}(4\text{ m})^2(3\text{ m})^2 = 18\text{ m}^4.$$

Determine the moments and products of inertia from Appendix B.

$$\tan 2\theta_p = \frac{2I_{xy}}{I_y - I_x} = \frac{2(18)}{48 - 9} = 0.923.$$

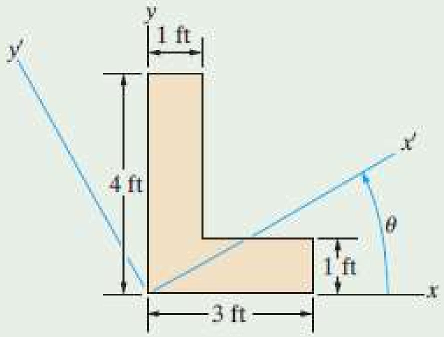
This yields $\theta_p = 21.4°$.

Determine θ_p from Eq. (8.26).



$$
\left.
\begin{aligned}
I_{x'} &= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta, \\
&= \left(\frac{9+48}{2}\right) + \left(\frac{9-48}{2}\right)\cos[2(21.4°)] - (18)\sin[2(21.4°)] \\
&= 1.96\ \text{m}^4, \\
I_{y'} &= \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta, \\
&= \left(\frac{9+48}{2}\right) - \left(\frac{9-48}{2}\right)\cos[2(21.4°)] + (18)\sin[2(21.4°)] \\
&= 55.0\ \text{m}^4,
\end{aligned}
\right\}
$$

The moments of inertia of the area in terms of the xy coordinate system shown are $I_x = 22\ \text{ft}^4$, $I_y = 10\ \text{ft}^4$, and $I_{xy} = 6\ \text{ft}^4$. (a) Determine $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ for $\theta = 30°$. (b) Determine a set of principal axes and the corresponding principal moments of inertia.



$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$= \left(\frac{22 + 10}{2}\right) + \left(\frac{22 - 10}{2}\right)\cos[2(30°)] - (6)\sin[2(30°)] = 13.8\ \text{ft}^4,$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$= \left(\frac{22 + 10}{2}\right) - \left(\frac{22 - 10}{2}\right)\cos[2(30°)] + (6)\sin[2(30°)] = 18.2\ \text{ft}^4,$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$$= \left(\frac{22 - 10}{2}\right)\sin[2(30°)] + (6)\cos[2(30°)] = 8.2\ \text{ft}^4.$$

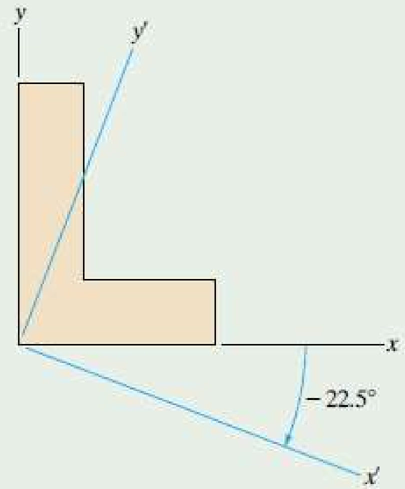
(b) Determine θ_p We substitute the moments of inertia in terms of the *xy* coordinate system into Eq. (8.26), yielding

$$\tan 2\theta_p = \frac{2I_{xy}}{I_y - I_x} = \frac{2(6)}{10 - 22} = -1.$$

Thus, $\theta_p = -22.5°$. The principal axes corresponding to this value of θ_p are shown in Fig. a.

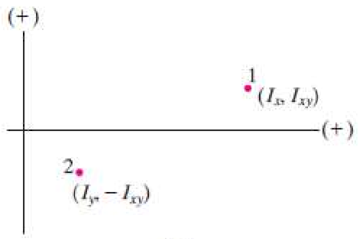
Calculate $I_{x'}$ and $I_{y'}$ We substitute $\theta_p = -22.5°$ into Eqs. (8.23) and (8.24), obtaining the principal moments of inertia:

$$I_{x'} = 24.5 \text{ ft}^4, \quad I_{y'} = 7.5 \text{ ft}^4.$$

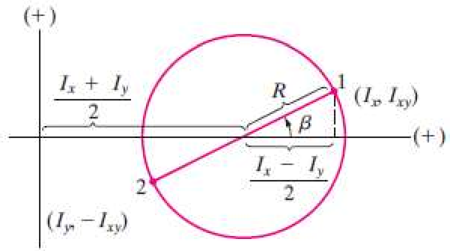


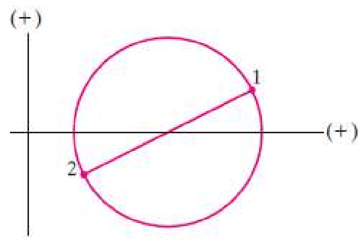
(a) The set of principal axes corresponding to $\theta_p = -22.5°$.

Circunferencia de Mohr



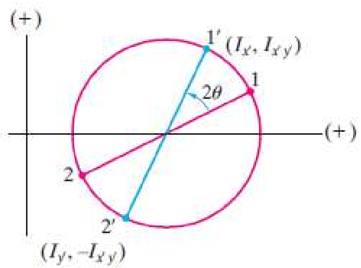
(a)

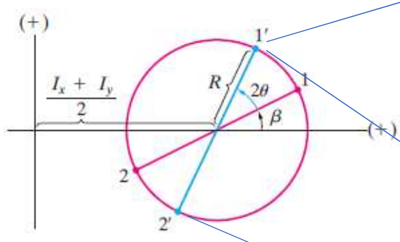




(b)

$$\sin \beta = \frac{I_{xy}}{R}, \qquad \cos \beta = \frac{I_x - I_y}{2R}, \qquad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2}.$$





$$\frac{I_x + I_y}{2} + R\cos(\beta + 2\theta)$$

$$= \frac{I_x + I_y}{2} + R(\cos\beta\cos 2\theta - \sin\beta\sin 2\theta)$$

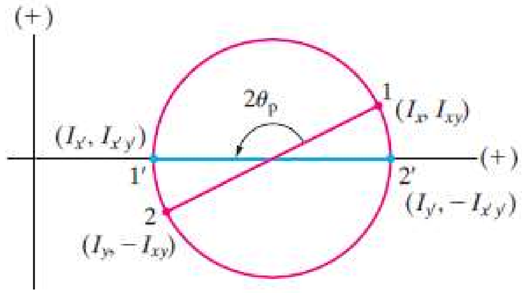
$$= \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta = I_{x'}.$$

$$R\sin(\beta + 2\theta) = R(\sin\beta\cos 2\theta + \cos\beta\sin 2\theta)$$

$$= I_{xy}\cos 2\theta + \frac{I_x - I_y}{2}\sin 2\theta = I_{x'y'},$$

$$-R\sin(\beta + 2\theta) = -I_{x'y'}.$$

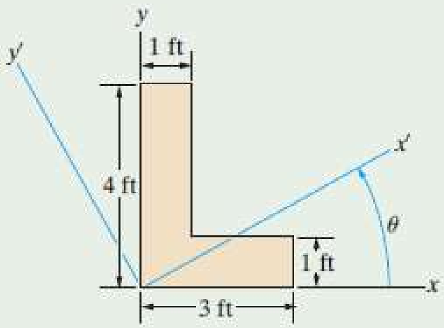
Circunferencia de Mohr
Momentos de Inercia Principales

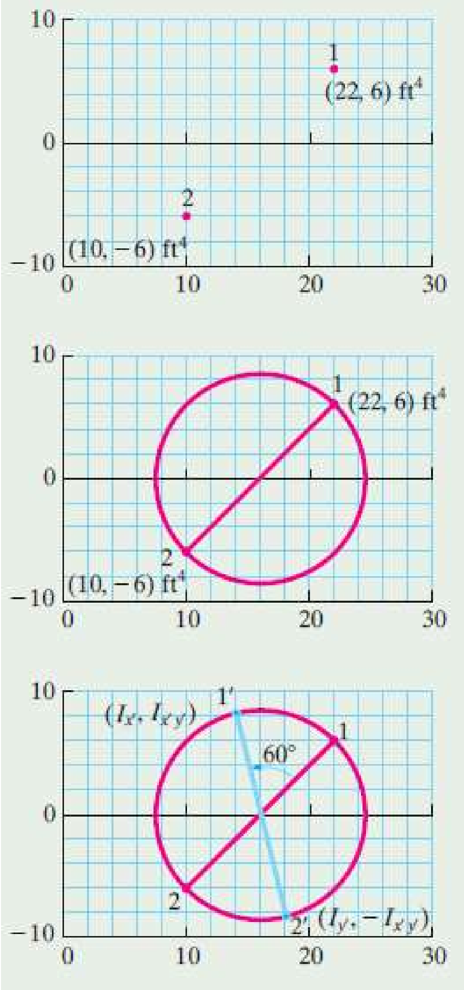


$$\text{Principal moments of inertia} = \frac{I_x + I_y}{2} \pm R$$

$$= \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + (I_{xy})^2}.$$

The moments and product of inertia of the area in terms of the xy coordinate system are $I_x = 22\ \text{ft}^4$, $I_y = 10\ \text{ft}^4$, and $I_{xy} = 6\ \text{ft}^4$. Use Mohr's circle to determine the moments of inertia $I_{x'}$, $I_{y'}$, and $I_{x'y'}$ for $\theta = 30°$.





Draw a straight line through the center of the circle at an angle $2\theta = 60°$ measured counterclockwise from point 1. From the coordinates of points 1′ and 2′, $I_{x'} = 14\ \text{ft}^4$, $I_{y'} = 18\ \text{ft}^4$, and $I_{x'y'} = 8\ \text{ft}^4$.

FIN